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THE LOGIC OF RELATIVES.

§ 1. *Three Grades of Clearness.*—The third volume of Professor Schröder's *Exact Logic*,[1] which volume bears separately the title I have chosen for this paper, is exciting some interest even in this country. There are in America a few inquirers into logic, sincere and diligent, who are not of the genus that buries its head in the sand,—men who devote their thoughts to the study with a view to learning something that they do not yet know, and not for the sake of upholding orthodoxy, or any other foregone conclusion. For them this article is written as a kind of popular exposition of the work that is now being done in the field of logic. To them I desire to convey some idea of what the new logic is, how two "algebras," that is, systems of diagrammatical representation by means of letters and other characters, more or less analogous to those of the algebra of arithmetic, have been invented for the study of the logic of relatives, and how Schröder uses one of these (with some aid from the other and from other notations) to solve some interesting problems of reasoning. I also wish to illustrate one other of several important uses to which the new logic may be put. To this end I must first clearly show what a relation is.

Now there are three grades of clearness in our apprehensions of the meanings of words. The first consists in the connexion of

[1] *Algebra und Logik der Relative.* Leipsic: B. G. Teubner. 1895. Price, 16 M.

the word with familiar experience. In that sense, we all have a clear idea of what *reality* is and what *force* is,—even those who talk so glibly of mental force being correlated with the physical forces. The second grade consists in the abstract definition, depending upon an analysis of just what it is that makes the word applicable. An example of defective apprehension in this grade is Professor Tait's holding (in an appendix to the reprint of his Britannica article, *Mechanics*) that energy is "objective" (meaning it is a substance), because it is permanent, or "persistent." For independence of time does not of itself suffice to make a substance; it is also requisite that the aggregant parts should always preserve their identity, which is not the case in the transformations of energy. The third grade of clearness consists in such a representation of the idea that fruitful reasoning can be made to turn upon it, and that it can be applied to the resolution of difficult practical problems.

§ 2. *Of the term Relation in its first Grade of Clearness.*—An essential part of speech, the Preposition, exists for the purpose of expressing relations. Essential it is, in that no language can exist without prepositions, either as separate words placed before or after their objects, as case-declensions, as syntactical arrangements of words, or some equivalent forms. Such words as "brother," "slayer," "at the time," "alongside," "not," "characteristic property" are relational words, or *relatives*, in this sense, that each of them *becomes a general name when another general name is affixed to it as object.* In the Indo-European languages, in Greek, for example, the so-called genitive case (an inapt phrase like most of the terminology of grammar) is, very roughly speaking, the form most proper to the attached name. By such attachments, we get such names as "brother *of* Napoleon," "slayer *of* giants," "*ἐπὶ Ἐλλισσαίου*, at the time of Elias," "*παρὰ ἀλλήλων*, alongside *of* each other," "not guilty," "a characteristic property *of* gallium." *Not* is a relative because it means "other than"; *scarcely*, though a relational word of highly complex meaning, is not a relative. It has, however, to be treated in the logic of relatives. Other relatives do not become general names until two or more names have been thus affixed. Thus, "giver to the city" is just such a relative as the preceding; for "giver to the city of a statue of himself" is a complete general name (that is, there might be several such humble admirers of themselves, though there be but one, as yet); but "giver" requires *two* names to be attached to it, before it becomes a complete name. The dative case is a somewhat usual form for the second object. The archaic instrumental and locative cases were serviceable for third and fourth objects.

Our European languages are peculiar in their marked differentiation of common nouns from verbs. *Proper* nouns must exist in all languages; and so must such "pronouns," or indicative words, as *this*, *that*, *something*, *anything*. But it is probably true that in the great majority of the tongues of men, distinctive common nouns either do not exist or are exceptional formations. In their meaning as they stand in sentences, and in many comparatively widely-studied languages, common nouns are akin to participles, as being mere inflexions of verbs. If a language has a verb meaning "is a man," a noun "man" becomes a superfluity. For all men are mortals is perfectly expressed by "Anything either is-a-man not or is-a-mortal." Some man is a miser is expressed by "Something both is-a-man and is-a-miser." The best treatment of the logic of relatives, as I contend, will dispense altogether with class names and only use such verbs. A verb requiring an object or objects to complete the sense may be called a *complete relative*.

A verb by itself signifies a mere dream, an imagination unattached to any particular occasion. It calls up in the mind an *icon*. A *relative* is just that, an icon, or image, without attachments to experience, without "a local habitation and a name," but with indications of the need of such attachments.

An indexical word, such as a proper noun or demonstrative or selective pronoun, has force to draw the attention of the listener to some hecceity common to the experience of speaker and listener. By a hecceity, I mean, some element of existence which, not merely by the likeness between its different apparitions, but by an inward force of identity, manifesting itself in the continuity of its apparition throughout time and in space, is distinct from everything else, and is thus fit (as it can in no other way be) to receive a proper name or to be indicated as *this* or *that*. Contrast this with the signification of the verb, which is sometimes in my thought, sometimes in yours, and which has no other identity than the agreement between its several manifestations. That is what we call an abstraction or idea. The nominalists say it is a *mere* name. Strike out the "mere," and this opinion is approximately true. The realists say it *is* real. Substitute for "is," *may be*, that is, *is* provided experience and reason shall, as their final upshot, uphold the truth of the particular predicate, and the natural existence of the law it expresses, and this is likewise true. It is certainly a great mistake to look upon an idea, merely because it has not the mode of existence of a hecceity, as a lifeless thing.

The proposition, or sentence, signifies that an eternal fitness, or truth, a permanent conditional force, or law, attaches certain hecceities to certain parts of an idea. Thus, take the idea of "buying by—of—from—in exchange for—." This has four places where hecceities, denoted by indexical words, may be attached. The proposition "A buys B from C at the price D," signifies an eternal, irrefragable, conditional force gradually compelling those attachments in the opinions of inquiring minds.

Whether or not there be in the reality any definite separation between the hecceity-element and the idea-element is a question of metaphysics, not of logic. But it is certain that in the expression of a fact we have a considerable range of choice as to how much we will denote by the indexical and how much signify by iconic words. Thus, we have stated "all men are mortal" in such a form that there is but one index. But we may also state it thus: "Taking anything, either it possesses not humanity or it possesses mortality." Here "humanity" and "mortality" are really proper names, or purely denotative signs, of familiar ideas. Accordingly, as here stated, there are three indices. Mathematical reasoning largely depends on this treatment of ideas as things; for it aids in the iconic representation of the whole fact. Yet for some purposes it is disadvantageous. These truths will find illustration in § 13 below.

Any portion of a proposition expressing ideas but requiring something to be attached to it in order to complete the sense, is in a general way relational. But it is only a *relative* in case the attachment of indexical signs will suffice to make it a proposition, or, at least, a complete general name. Such a word as *exceedingly* or *previously* is relational, but is not a relative, because significant words require to be added to it to make complete sense.

§ 3. *Of Relation in the Second Grade of Clearness.*—Is relation anything more than a connexion between two things? For example, can we not state that A gives B to C without using any other relational phrase than that one thing is connected with another? Let us try. We have the general idea of *giving*. Connected with it are the general ideas of *giver*, *gift*, and "*donee*." We have also a particular transaction connected with no general idea except through that of giving. We have a first party connected with this transaction and also with the general idea of giver. We have a second party connected with that transaction, and also with the general idea of "donee." We have a subject connected with that transaction and also with the general idea of gift. A is the only hecceity directly connected with the first party; C is the only hecceity directly connected with the second party, B is the only hecceity directly connected with the subject. Does not this long statement amount to this, that A gives B to C?

In order to have a distinct conception of Relation, it is necessary not merely to answer this question but to comprehend the reason of the answer. I shall answer it in the negative. For, in the first place, if relation were nothing but connexion of two things, all things would be connected. For certainly, if we say that A is unconnected with B, that non-connexion is a relation between A and B. Besides, it is evident that any two things whatever make a pair. Everything, then, is equally related to everything else, if mere connexion be all there is in relation. But that which is equally and necessarily true of everything is no positive fact, at all. This would reduce relation, considered as simple connexion between two things, to nothing, unless we take refuge in saying that relation *in general* is indeed nothing, but that *modes* of relation are something. If, however, these different modes of relation are different modes of connexion, relation ceases to be simple bare connexion. Going back, however, to the example of the last paragraph, it will be pointed out that the peculiarity of the mode of connexion of A with the transaction consists in A's being in connexion with an element connected with the transaction, which element is connected with the peculiar general idea of a *giver*. It will, therefore, be said, by those who attempt to defend an affirmative answer to our question, that the peculiarity of a mode of connexion consists in this, that that connexion is indirect and takes place through something which is connected with a peculiar general idea. But I say that is no answer at all; for if all things are equally connected, nothing can be more connected with one idea than with another. This is unanswerable. Still, the affirmative side may modify their position somewhat. They may say, we grant that it is necessary to recognise that relation is something more than connexion; it is *positive* connexion. Granting that all things are connected, still all are not positively connected. The various modes of relationship are, then, explained as above. But to this I reply: you propose to make the peculiarity of the connexion of A with the transaction depend (no matter by what machinery) upon that connexion having a positive connexion with the idea of a giver. But "positive connexion" is not enough; the relation of the general idea is quite peculiar. In order that it may be characterised, it must, on your principles, be made indirect, taking place through something which is itself connected with a general idea. But this last connexion is again more than a mere general positive connexion. The same device must be resorted to, and so on *ad infinitum*. In short, you are guilty of a *circulus in definiendo*. You make the relation of any two things consist in their connexion being connected with a general idea. But that last connexion is, on your own principles, itself a *relation*, and you are thus defining relation by relation; and if for the second occurrence you substitute the definition, you have to repeat the substitution *ad infinitum*.

The affirmative position has consequently again to be modified. But, instead of further tracing possible tergiversations, let us directly establish one or two positive positions. In the first place, I say that every relationship concerns some definite number of correlates. Some relations have such properties that this fact is concealed. Thus, any number of men may be brothers. Still, brotherhood is a relation between pairs. If A, B, and C are all brothers, this is merely the consequence of the three relations, A is brother of B, B is brother of C, C is brother of A. Try to construct a relation which shall exist either between two or between three things such as "—is either a brother or betrayer of—to—." You can only make sense of it by somehow interpreting the dual relation as a triple one. We may express this as saying that every relation has a definite number of blanks to be filled by indices, or otherwise. In the case of the majority of relatives, these blanks are qualitatively different from one another. These qualities are thereby communicated to the connexions.

In a complete proposition there are no blanks. It may be called a *medad*, or *medadic relative*, from μηδαμός, none, and -άδα the accusative ending of such words as μόνας, δυάς, τριάς, τετράς, etc.[1] A non-relative name with a substantive verb, as "—is a man," or "man that is—," or "—'s manhood" has one blank; it is a *monad*, or *monadic relative*. An ordinary relative with an active verb as "—is a lover of—" or "the loving by—of—" has two blanks; it is a *dyad*, or *dyadic relative*. A higher relative similarly treated has a plurality of blanks. It may be called a *polyad*. The rank of a relative among these may be called its *adinity*, that is, the peculiar quality of the number it embodies.

A *relative*, then, may be defined as the equivalent of a word or phrase which, either as it is (when I term it a *complete* relative), or else when the verb "is" is attached to it (and if it wants such attachment, I term it a *nominal* relative), becomes a sentence with some number of proper names left blank. A *relationship*, or *fundamentum relationis*, is a fact relative to a number of objects, consid-

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[1] The Pythagoreans, who seem first to have used these words, probably attached a patronymic signification to the termination. A *triad* was derivative of *three*, etc.

ered apart from those objects, as if, after the statement of the fact, the designations of those objects had been erased. A *relation* is a relationship considered as something that may be said to be true of one of the objects, the others being separated from the relationship yet kept in view. Thus, for each relationship there are as many relations as there are blanks. For example, corresponding to the relationship which consists in one thing loving another there are two relations, that of loving and that of being loved by. There is a nominal relative for each of these relations, as "lover of—," and "loved by—." These nominal relatives belonging to one relationship, are in their relation to one another termed *correlatives.* In the case of a dyad, the two correlatives, and the corresponding relations are said, each to be the *converse* of the other. The objects whose designations fill the blanks of a complete relative are called the *correlates.* The correlate to which a nominal relative is attributed is called the *relate.*

In the statement of a relationship, the designations of the correlates ought to be considered as so many *logical subjects* and the relative itself as the *predicate.* The entire set of logical subjects may also be considered as a *collective subject,* of which the statement of the relationship is *predicate.*

§ 4. *Of Relation in the third Grade of Clearness.*—Mr. A. B. Kempe has published in the *Philosophical Transactions* a profound and masterly "Memoir on the Theory of Mathematical Form," which treats of the representation of relationships by "Graphs," which is Clifford's name for a diagram, consisting of spots and lines, in imitation of the chemical diagrams showing the constitution of compounds. Mr. Kempe seems to consider a relationship to be nothing but a complex of bare connexions of pairs of objects, the opinion refuted in the last section. Accordingly, while I have learned much from the study of his memoir, I am obliged to modify what I have found there so much that it will not be convenient to cite it; because long explanations of the relation of my views to his would become necessary if I did so.

A chemical atom is quite like a relative in having a definite number of loose ends or "unsaturated bonds," corresponding to the blanks of the relative. In a chemical molecule, each loose end of one atom is joined to a loose end, which it is assumed must belong to some other atom, although in the vapor of mercury, in argon, etc., two loose ends of the same atom would seem to be joined; and why pronounce such hermaphrodism impossible? Thus the chemical molecule is a *medad*, like a complete proposition. Regarding proper names and other indices, after an "is" has been attached to them, as monads, they, together with other monads, correspond to the two series of chemical elements, H, Li, Na, K, Rb, Cs, etc., and Fl, Cl, Br, I. The dyadic relatives correspond to the two series, Mg, Ca, Sr, Ba, etc., and O, S, Se, Te, etc. The triadic relatives correspond to the two series B, Al, Zn, In, Tl, etc., and N, P, As, Sb, Bi, etc. Tetradic relatives are, as we shall see, a superfluity; they correspond to the series C, Si, Ti, Sn, Ta, etc. The proposition "John gives John to John" corresponds in

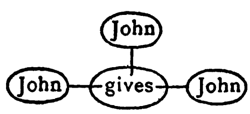

Fig. 1.

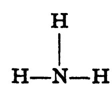

Fig. 2.

its constitution, as Figs. 1 and 2 show, precisely to ammonia.

But beyond this point the analogy ceases to be striking. In fact, the analogy with the ruling theory of chemical compounds quite breaks down. Yet I cannot resist the temptation to pursue it. After all, any analogy, however fanciful, which serves to focus attention upon matters which might otherwise escape observation is valuable. A chemical compound might be expected to be quite as much like a proposition as like an algebraical invariant; and the brooding upon chemical graphs has hatched out an important theory in invariants. Fifty years ago, when I was first studying chemistry, the theory was that every compound consisted of two oppositely electrified atoms or radicles; and in like manner every compound radicle consisted of two opposite atoms or radicles. The argument to this effect was that chemical attraction is evidently between things unlike one another and evidently has a saturation point; and further that we observe that it is the elements the most extremely unlike which attract one another. Lothar Meyer's curve having for its ordinates the atomic volumes of the elements and for its abscissas their atomic weights tends to support the opinion that elements strongly to attract one another must have opposite characters; for we see that it is the elements on the steepest downward slopes of that curve which have the strongest attractions for the elements on the steepest upward inclines. But when chemists became convinced of the doctrine of valency, that is, that every element has a fixed number of loose ends, and when they consequently began to write graphs for compounds, it seems to have been assumed that this necessitated an abandonment of the position that atoms and radicles combine by opposition of characters, which had further been weakened by the refutation of some mistaken arguments in its favor. But if chemistry is of no aid to logic, logic here comes in to enlighten chemistry. For in logic, the medad must always be composed of one part having a negative, or antecedental, character, and another part of a positive, or consequental, character; and if either of these parts is compound its constituents are similarly related to one another. Yet this does not, at all, interfere with the doctrine that each relative has a definite number of blanks or loose ends. We shall find that, in logic, the negative character is a character of reversion in this sense, that if the negative part of a medad is compound, *its* negative part has, on the whole, a positive character. We shall also find, that if the negative part of a medad is compound, the bond joining its positive and negative parts has its character reversed, just as those relatives themselves have.

Several propositions are in this last paragraph stated about logical medads which now must be shown to be true. In the first place, although it be granted that every relative has a definite number of blanks, or loose ends, yet it would seem, at first sight, that there is no need of each of these joining no more than one other. For instance, taking the triad "—kills—to gratify—," why may not the three loose ends all join in one node and then be connected with the loose end of the monad "John is—" as in Fig. 3 making the proposition "John it is that kills what is John to gratify what is John"? The answer is, that a little exercise of generalising power will show that such a four-way node is really a tetradic relative,

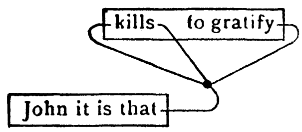

Fig. 3.

which may be expressed in words thus, "—is identical with—and with—and with—"; so that the medad is really equivalent to that

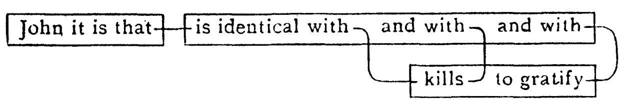

Fig. 4.

of Fig. 4, which corresponds to prussic acid as shown in Fig. 5.

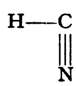

Fig. 5.

Thus, it becomes plain that every node of bonds is equivalent to a relative; and the doctrine of valency is established for us in logic.

We have next to inquire into the proposition that in every combination of relatives there is a negative and a positive constituent. This is a corollary from the general logical doctrine of the illative character of the copula, a doctrine precisely opposed to the opinion of the quantification of the predicate. A satisfactory discussion of this fundamental question would require a whole article. I will only say in outline that it can be positively demonstrated in several ways that a proposition of the form "man = rational animal," is a compound of propositions each of a form which may be stated thus: "*Every* man (if there be any) is a rational animal" or "Men are *exclusively* (if anything) rational animals." Moreover, it must be acknowledged that the illative relation (that expressed by "therefore") is the most important of logical relations, the be-all and the end-all of the rest. It can be demonstrated that formal logic needs no other elementary logical relation than this;

but that with a symbol for this and symbols of relatives, including monads, and with a mode of representing the attachments of them, all syllogistic may be developed, far more perfectly than any advocate of the quantified predicate ever developed it, and in short in a way which leaves nothing to be desired. This in fact *will* be virtually shown in the present paper. It can further be shown that no other copula will of itself suffice for all purposes. Consequently, the copula of equality ought to be regarded as merely derivative.

Now, in studying the logic of relatives we must sedulously avoid the error of regarding it as a highly specialised doctrine. It is, on the contrary, nothing but formal logic generalised to the very tiptop. In accordance with this view, or rather with this theorem (for it is susceptible of positive demonstration), we must regard the *relative copula*, which is the bond between two blanks of relatives, as only a generalisation of the ordinary copula, and thus of the "*ergo.*" When we say that from the proposition A the proposition B necessarily follows, we say that "the truth of A in *every way* in which it can exist at all is the truth of B," or otherwise stated "A is true *only* in so far as B is true." This is the very same relation which we express when we say that "*every* man is mortal," or "men are *exclusively* mortals." For this is the same as to say, "Take anything whatever, M; then, if M is a man, it follows necessarily that M is mortal." This mode of junction is essentially the same as that between the relatives in the compound relative "lover, in *every* way in which it may be a lover at all, of a servant," or, otherwise expressed, "lover (if at all) *exclusively* of servants." For to say that "Tom is a lover (if at all) only of servants of Dick," is the same as to say "Take anything whatever, M; then, if M is loved by Tom, M is a servant of Dick," or "everything there may be that is loved by Tom is a servant of Dick."

Now it is to be observed that the illative relation is not simply convertible; that is to say, that "from A necessarily follows B" does not necessarily imply that "from B necessarily follows A." Among the vagaries of some German logicians of some of the inexact schools, the convertibility of illation (like almost every other imaginable absurdity) has been maintained; but all the other inexact schools deny it, and exact logic condemns it, at once. Consequently, the copula of inclusion, which is but the *ergo* freed from the accident of asserting the truth of its antecedent, is equally inconvertible. For though "men include only mortals," it does not follow that "mortals include *only* men," but, on the contrary, what follows is "mortals include *all* men." Consequently, again, the fundamental *relative copula* is inconvertible. That is, because "Tom loves (if anybody) only a servant (or servants) of Dick," it does not follow that "Dick is served (if at all) only by somebody loved by Tom," but, on the contrary, what follows is "Dick is master of *every* person (there may be) who is loved by Tom." We thus see clearly, first, that, as the fundamental relative copula, we must take that particular mode of junction; secondly, that that mode is at bottom the mode of junction of the *ergo*, and so joins a relative of antecedental character to a relative of consequental character; and, thirdly, that that copula is inconvertible, so that the two kinds of constituents are of opposite characters. There are, no doubt, convertible modes of junction of relatives, as in "lover of a servant;[1] but it will be shown below that these are complex and indirect in their constitution.

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[1] Professor Schröder proposes to substitute the word "symmetry" for *convertibility*, and to speak of *simply convertible* modes of junction as "symmetrical.' Such an example of wanton disregard of the admirable traditional terminology of logic, were it widely followed, would result in utter uncertainty as to what any

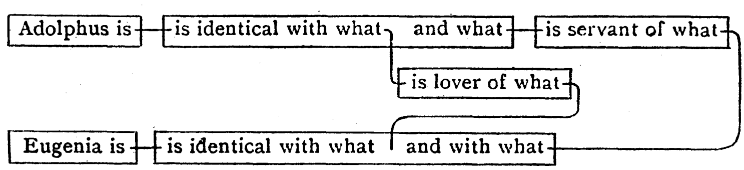

Fig. 6.

writer on logic might mean to say, and would thus be utterly fatal to all our efforts to render logic exact. Professor Schröder denies that the mode of junction in "lover of a servant" is "symmetrical," which word in practice he makes synonymous with "commutative," applying it only to such junctions as that between "lover" and "servant" in "Adolphus is at once lover and servant of Eugenia." Commutativity depends on one or more polyadic relatives having two like blanks as shown in Fig. 6.

It remains to be shown that the antecedent part of a medad has a negative, or reversed, character, and how this, in case it be compound, affects both its relatives and their bonds. But since this matter is best studied in examples, I will first explain how I propose to draw the logical graphs.

It is necessary to use, as the sign of the relative copula, some symbol which shall distinguish the antecedent from the consequent; and since, if the antecedent is compound (owing to the very character which I am about to demonstrate, namely, its reversing the characters of the relatives and the bonds it contains), it is very important to know just how much is included in that antecedent, while it is a matter of comparative indifference how much is included in the consequent (though it is simply everything not in the antecedent), and since further (for the same reason) it is important to know how many antecedents, each after the first a part of another, contain a given relative or copula, I find it best to make the line which joins antecedent and consequent encircle the whole of the former. Letters of the alphabet may be used as abbreviations of complete relatives; and the proper number of bonds may be attached to each. If one of these is encircled, that circle must have a bond corresponding to each bond of the encircled letter. Chemists sometimes write above atoms Roman numerals to indicate their *adinities;* but I do not think this necessary. Fig. 7 shows, in a com-

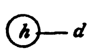

Fig. 7.

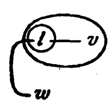

Fig. 8.

plete medad, my sign of the relative copula. Here, *h* is the monad "—is a man," and *d* is the monad "—is mortal." The antecedent is completely enclosed, and the meaning is "Anything whatever, if it be a man, is mortal." If the circle encloses a dyadic or polyadic relative, it must, of course, have a tail for every bond of that relative. Thus, in Fig. 8, *l* is the dyad "—loves—," and it is important to remark that the bond to the left is the lover and that to the right is the loved. Monads are the only relatives for which we need not be attentive to the positions of attachment of the bonds. In this figure, *w* is the monad "—is wise," and *v* is the monad "—is virtuous." The *l* and *v* are enclosed in a large common circle. Had this not been done, the medad could not be read (as far as any rules yet given show), because it would not consist of antecedent and consequent. As it is, we begin the reading of the medad at the bond connecting antecedent and consequent. Every bond of a logical graph denotes a hecceity; and every unencircled bond (as this one is) stands for any hecceity the reader may choose from the universe. This medad evidently refers to the universe of men. Hence the interpretation begins: "Let M be any man you please." We proceed along this bond in the direction of the antecedent, and on entering the circle of the antecedent we say: "If M be." We then enter the inner circle. Now, entering a circle means a relation to *every*. Accordingly we add "whatever." Traversing *l* from left to right, we say "lover." (Had it been from right to left we should have read it "loved.") Leaving the circle is the mark of a relation "only to," which words we add. Coming to *v* we say "what is virtuous." Thus our antecedent reads: "Let M be any man you please. If M be whatever it may that is lover only to the virtuous." We now return to the consequent and read, "M is wise." Thus the whole means, "Whoever loves only the virtuous is wise."

As another example, take the graph of Fig. 9, where *l* has the

Fig. 9.

same meaning as before and *m* is the dyad "—is mother of—." Suppose we start with the left hand bond. We begin with saying "Whatever." Since cutting this bond does not sever the medad, we proceed at once to read the whole as an unconditional statement and we add to our "whatever" "there is." We can now move round the ring of the medad either clockwise or counter-clockwise. Taking the last way, we come to *l* from the left hand and therefore add "is a lover." Moving on, we enter the circle round *m;* and entering a circle is a sign that we must say "of *every thing* that." Since we pass through *m* backwards we do not read "is mother" but "is mothered" or "has for mother." Then, since we pass *out* of the circle we should have to add "only"; but coming back, as we do, to the starting point, we need only say "that same thing." Thus, the interpretation is "Whatever there is, is lover of everything that has for mother that same thing," or "Every woman loves everything of which she is mother." Starting at the same point and going round the other way, the reading would be "Everybody is mother (if at all) only of what is loved by herself." Starting on the right and proceeding clockwise, "Everything is loved by every mother of itself." Proceeding counter-clockwise, "Everything has for mothers only lovers of itself."

Triple relatives afford no particular difficulty. Thus, in Fig. 10, *w* and *v* have the same significations as before; *r* is the monad, "—is a reward," and *g* is the triad "—gives — to —." It can be read either

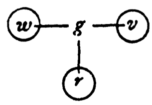

Fig. 10.

"Whatever is wise gives every reward to every virtuous person," or "Every virtuous person has every reward given to him by everybody that is wise," or "Every reward is given by everybody who is wise to every virtuous person."

A few more examples will be instructive. Fig. 11, where A is the proper name Alexander means "Alexander loves only the virtuous," i. e., "Take anybody you please; then, if he be Alexander and if he loves anybody, this latter is virtuous."

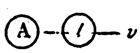

Fig. 11.

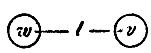

Fig. 12.

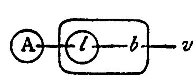

Fig. 13.

If you attempt, in reading this medad, to start to the right of *l*, you fall into difficulty, because your antecedent does not then consist of an antecedent and consequent, but of two circles joined by a bond, a combination to be considered below. But Fig. 12 may be read with equal ease on whichever side of *l* you begin, whether as "whoever is wise loves everybody that is virtuous," or "whoever is virtuous is loved by everybody that is wise." If in Fig. 13

-b- be the dyad "—is a benefactor of—," the medad reads, "Alexander stands only to virtuous persons in the relation of loving only their benefactors."

Fig. 14, where *-s-* is the dyad "—is a servant of —" may be read, according to the above principles, in the several ways following:

"Whoever stands to any person in the relation of lover to none but his servants benefits him."

"Every person stands only to a person benefited by him in the relation of a lover only of a servant of that person."

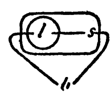

Fig. 14.

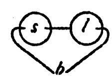

Fig. 15.

"Every person, M, is benefactor of everybody who stands to M in the relation of being served by everybody loved by him."

"Every person, N, is benefited by everybody who stands to N in the relation of loving only servants of him."

"Every person, N, stands only to a benefactor of N in the relation of being served by everybody loved by him."

"Take any two persons, M and N. If, then, N is served by every lover of M, N is benefited by M."

Fig. 15 represents a medad which means, "Every servant of any person, is a benefactor of whomever may be loved by that person." Equivalent statements easily read off from the graphs are as follows:

"Anybody, M, no matter who, is servant (if at all) only of somebody who loves (if at all) only persons benefited by M."

"Anybody, no matter who, stands to every master of him in the relation of benefactor of whatever person may be loved by him."

"Anybody, no matter who, stands to whoever loves him in the relation of being benefited by whatever servant he may have."

"Anybody, N, is loved (if at all) only by a person who is served (if at all) only by benefactors of N."

"Anybody, no matter who, loves (if at all) only persons benefited by all servants of his."

"Anybody, no matter who, is served (if at all) only by benefactors of everybody loved by him."

I will now give an example containing triadic relatives, but no monads. Let *p* be "—prevents—from communicating with—," the second blank being represented by a bond from the right of *p* and the third by a bond from below *p*. Let *β* mean "—would betray—to—," the arrangement of bonds being the same as with *p*. Then, Fig. 16 means that "whoever loves only persons who prevent every servant of any person, A, from communicating with any person, B, would betray B to A." I will only notice one equivalent statement, viz.: "Take any three persons, A, B, C, no matter who. Then, either C betrays B to A, or else two persons, M and N, can be found, such that M does not prevent N from communicating with B, although M is loved by C and N is a servant of A."

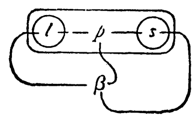

Fig. 16.

This last interpretation is an example of the method which is, by far, the plainest and most unmistakable of any in complicated cases. The rule for producing it is as follows:

1. Assign a letter of the alphabet to denote the hecceity represented by each bond.[1]

2. Begin by saying: "Take any things you please, namely," and name the letters representing bonds not encircled; then add, "Then suitably select objects, namely," and name the letters representing bonds each once encircled; then add, "Then take any things you please, namely," and name the letters representing bonds each twice encircled. Proceed in this way until all the letters

[1] In my method of graphs, the spots represent the relatives, their bonds the hecceities; while in Mr. Kempe's method, the spots represent the objects, whether individuals or abstract ideas, while their bonds represent the relations. Hence, my own exclusive employment of bonds between pairs of spots does not, in the least, conflict with my argument that in Mr. Kempe's method such bonds are insufficient.

representing bonds have been named, no letter being named until all those encircled fewer times have been named; and each hecceity corresponding to a letter encircled odd times is to be suitably chosen according to the intent of the assertor of the medad proposition, while each hecceity corresponding to a bond encircled even times is to be taken as the interpreter or the opponent of the proposition pleases.

3. Declare that you are about to make statements concerning certain propositions, to which, for the sake of convenience, you will assign numbers in advance of enunciating them or stating their relations to one another. These numbers are to be formed in the following way. There is to be a number for each letter of the medad (that is for those which form spots of the graph, not for the letters assigned by clause 1 of this rule to the bonds), and also a number for each circle round more than one letter; and the first figure of that number is to be a 1 or a 2, according as the letter or the circle is in the principal antecedent or the principal consequent; the second figure is to be 1 or 2, according as the letter or the circle belongs to the antecedent or the consequent of the principal antecedent or consequent, and so on.

Declare that one or other of those propositions whose numbers contain no 1 before the last figure is true. Declare that each of those propositions whose numbers contain an odd number of 1's before the last figure consists in the assertion that *some one* or another of the propositions whose numbers commence with its number is true. For example, 11 consists in the assertion that either 111 or 1121 or 1122 is true, supposing that these are the only propositions whose numbers commence with 11. Declare that each of those propositions whose numbers contain an even number of 1's (or none) before the last figure consists in the assertion that *every one of* the propositions whose numbers commence with its number is true. Thus, 12 consists in the assertion that 121, 1221, 1222 are all true, provided those are the only propositions whose numbers commence with 12. The process described in this clause will be abridged except in excessively complicated cases.

4. Finally, you are to enunciate all those numbered proposi-

tions which correspond to single letters. Namely, each proposition whose number contains an even number of 1's, will consist in affirming the relative of the spot-letter to which that number corresponds after filling each blank with that bond-letter which by clause 1 of this rule was assigned to the bond at that blank. But if the number of the proposition contains an odd number of 1's, the relative, with its blanks filled in the same way, is to be denied.

In order to illustrate this rule, I will restate the meanings of the medads of Figs. 7–16, in all the formality of the rule; although such formality is uncalled for and awkward, except in far more complicated cases.

Fig. 7. Let A be anything you please. There are two propositions, 1 and 2, one of which is true. Proposition 1 is, that A is not a man. Proposition 2 is, that A is mortal. More simply, Whatever A may be, either A is not a man or A is mortal.

Fig. 8. Let A be anybody you please. Then, I will find a person, B, so that either proposition 1 or proposition 2 shall be true. Proposition 1 asserts that both propositions 11 and 12 are true. Proposition 11 is that A loves B. Proposition 12 is that B is not virtuous. Proposition 2 is that A is wise. More simply, Take anybody, A, you please. Then, either A is wise, or else a person, B, can be found such that B is not virtuous and A loves B.

Fig. 9. Let A and B be any persons you please. Then, either proposition 1 or proposition 2 is true. Proposition 1 is that A is not a mother of B. Proposition 2 is that A loves B. More simply, whatever two persons A and B may be, either A is not a mother of B or A loves B.

Fig. 10. Let A, B, C be any three things you please. Then, one of the propositions numbered, 1, 21, 221, 222 is true. Proposition 1 is that A is not wise. Proposition 21 is that B is not a reward. Proposition 221 is that C is not virtuous. Proposition 222 is that A gives B to C. More simply, take any three things, A, B, C, you please. Then, either A is not wise, or B is not a reward, or C is not virtuous, or A gives B to C.

Fig. 11. Take any two persons, A and B, you please. Then, one of the propositions 1, 21, 22 is true. 1 is that A is not Alex-

ander. 21 is that A does not love B. Proposition 3 is that B is virtuous.

Fig. 12. Take any two persons, A and B. Then, one of the propositions 1, 21, 22 is true. 1 is that A is not wise. 21 is that B is not virtuous. 22 is that A loves B.

Fig. 13. Take any two persons, A and C. Then a person, B can be found such that one of the propositions 1, 21, 22 is true. Proposition 21 asserts that both 211 and 212 are true. Proposition 1 that A is not Alexander. Proposition 211 is that A loves B. Proposition 212 is that B does not benefit C. Proposition 22 is that C is virtuous. More simply, taking any two persons, A and C, either A is not Alexander, or C is virtuous, or there is some person, B, who is loved by A without benefiting C.

Fig. 14. Take any two persons, A and B, and I will then select a person C. Either proposition 1 or proposition 2 is true. Proposition 1 is that both 11 and 12 are true. Proposition 11 is that A loves C. Proposition 12 is that C is not a servant of B. Proposition 2 is that A benefits B. More simply, of any two persons, A and B, either A benefits the other, B, or else there is a person, C, who is loved by A but is not a servant of B.

Fig. 15. Take any three persons, A, B, C. Then one of the propositions 1, 21, 22 is true. 1 is that A is not a servant of B; 21 is that B is not a lover of C; 22 is that A benefits C.

Fig. 16. Take any three persons, A, B, C. Then I can so select D and E, that one of the propositions 1 or 2 is true. 1 is that 11 and 121 and 122 are all true. 11 is that A loves D, 121 is that E is a servant of C, 122 is that D does not prevent E from communicating with B. 2 is that A betrays B to C.

I have preferred to give these examples rather than fill my pages with a dry abstract demonstration of the correctness of the rule. If the reader requires such a proof, he can easily construct it. This rule makes evident the reversing effect of the encirclements, not only upon the "quality" of the relatives as affirmative or negative, but also upon the selection of the hecceities as performable by advocate or opponent of the proposition, as well as upon the conjunctions of the propositions as disjunctive or conjunctive, or (to avoid this absurd grammatical terminology) as alternative or simultaneous.

It is a curious example of the degree to which the thoughts of logicians have been tied down to the accidents of the particular language they happened to write (mostly Latin), that while they hold it for an axiom that two *nots* annul one another, it was left for me to say as late as 1867[1] that *some* in formal logic ought to be understood, and could be understood, so that *some-some* should mean *any*. I suppose that were ordinary speech of any authority as to the forms of logic, in the overwhelming majority of human tongues two negatives intensify one another. And it is plain that if "not" be conceived as less than anything, what is less than that is *a fortiori* not. On the other hand, although *some* is conceived in our languages as *more than none*, so that two "somes" intensify one another, yet what it ought to signify for the purposes of syllogistic is that, instead of the selection of the instance being left,—as it is, when we say "any man is not good,"—to the opponent of the proposition, when we say "some man is not good," this selection is transferred to the opponent's opponent, that is to the defender of the proposition. Repeat the some, and the selection goes to the opponent's opponent's opponent, that is, to the opponent again, and it becomes equivalent to *any*. In more formal statement, to say "Every man is mortal," or "Any man is mortal," is to say, "A man, as suitable as any to prove the proposition false, is mortal," while "Some man is mortal" is equivalent to "A man, as suitable as any to prove the proposition *not* false, is mortal." "Some-some man is mortal" is accordingly "A man, as suitable as any to prove the proposition *not not*-false, is mortal."

In like manner, encircled $2N+1$ times, a disjunctive conjunction of propositions becomes a copulative conjunction. Here, the case is altogether similar. Encircled even times, the statement is that some one (or more) of the propositions is true; encircled odd times, the statement is that any one of the propositions is true.

[1] "On the Natural Classification of Arguments." *Proceedings of the American Academy of Arts and Sciences.*

The negative of "lover of every servant" is "non-lover of some servant." The negative of "lover every way (that it is a lover) of a servant" is "lover some way of a non-servant."

The general nature of a relative and of a medad has now been made clear. At any rate, it will become so, if the reader carefully goes through with the explanations. We have not, however, as yet shown how every kind of proposition can be graphically expressed, nor under what conditions a medad is necessarily true. For that purpose it will be necessary to study certain special logical relatives.

§ 5. *Triads the primitive relatives.*—That out of triads all polyads can be constructed is made plain by Fig. 17.

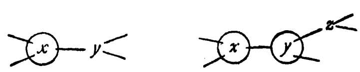

Fig. 17.

Fig. 18 shows that from two triads a dyad can be made. Fig. 19 shows that from one triad a monad can be made. Fig. 20 shows

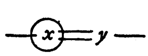

Fig. 18.

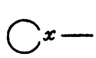

Fig. 19.

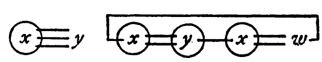

Fig. 20.

that from any even number of triads a medad can be made. In general, the union of a μ-ad and a ν-ad gives a $(\mu+\nu-2\lambda)$-ad, where λ is the number of bonds of union. This formula shows that *artiads*, or even-ads, can produce only *artiads*. But any perissid, or odd-ad (except a monad), can by repetition produce a relative of any *adinity*.

Since the principal object of a notation for relatives is not to produce a handy *calculus* for the solution of special logical problems, but to help the study of logical principles, the study of logical graphs from that point of view must be postponed to a future occasion. For present purposes that notation is best which carries analysis the furthest, and presents the smallest number of unanalyzed forms. It will be best, then, to use single letters for relatives of some one definite and odd number of blanks. We naturally choose three as the smallest number which will answer the purpose.

We shall, therefore, substitute for such a dyad as "—is lover of—" some such triad as "—is coexistent with | and a lover of—." If, then, we make -w- to signify "—is coexistent with | and with —," that which we have hitherto written as in Fig. 12 will be written as in Fig. 21. But having once recognised that such a mode

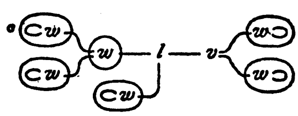

Fig. 21.

of writing is possible, we can continue to use our former methods, provided we now consider them as abbreviations.

The logical doctrine of this section, must, we may remark, find its application in metaphysics, if we are to accept the Kantian principle that metaphysical conceptions mirror those of formal logic.

§ 6. *Relatives of Second Intention.*—The general method of graphical representation of propositions has now been given in all its essential elements, except, of course, that we have not, as yet, studied any truths concerning special relatives; for to do so would seem, at first, to be "extralogical." Logic in this stage of its development may be called *paradisaical logic,* because it represents the state of Man's cognition before the Fall. For although, with this apparatus, it is easy to write propositions necessarily true, it is absolutely impossible to write any which is necessarily false, or, in any way which that stage of logic affords, to find out that anything is false. The mind has not as yet eaten of the fruit of the Tree of Knowledge of Truth and Falsity. Probably it will not be doubted that every child in its mental development necessarily passes through a stage in which he has some ideas, but yet has never recognised that an idea may be erroneous; and a stage that every child necessarily passes through must have been formerly passed through by the race in its adult development. It may be doubted whether many of the lower animals have any clear and steady conception of falsehood; for their instincts work so unerringly that there is little to force it upon their attention. Yet plainly without a knowledge of falsehood no development of discursive reason can take place.

This paradisaical logic appears in the study of non-relative formal logic. But *there* no possible avenue appears by which the knowledge of falsehood could be brought into this Garden of Eden except by the arbitrary and inexplicable introduction of the Serpent in the guise of a proposition necessarily false. The logic of relatives, affords such an avenue, and *that*, the very avenue by which in actual development, this stage of logic supervenes. It is the avenue of experience and logical reflexion.

By *logical* reflexion, I mean the observation of thoughts in their expressions. Aquinas remarked that this sort of reflexion is requisite to furnish us with those ideas which, from lack of contrast, ordinary external experience fails to bring into prominence. He called such ideas *second intentions.* It is by means of *relatives of second intention* that the general method of logical representation is to find completion.

Let ⋉ signify that "—is $\left\{\begin{matrix}\text{neither—}\\ \text{nor—}\end{matrix}\right.$." Then Fig. 22 means

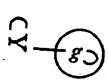

Fig. 22.

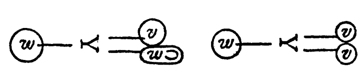

Fig. 23.

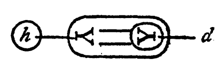

Fig. 24.

that taking any two things whatever, either the one is neither itself nor the other (putting it out of the question as an absurdity), or the other is a non-giver of something to that thing. That is, nothing gives all things, each to itself. Thus, the existence of any gen-

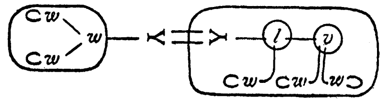

Fig. 25.

eral description of thing can be denied. Either medad of Fig. 23 means no wise men are virtuous. Fig. 24 is equivalent to Fig. 7. Fig. 25 means "each wise man is a lover of something virtuous."

Thus we see that this mode of junction,—lover of some virtuous,—which seems so simple,—is really complex. Fig. 26 means "some

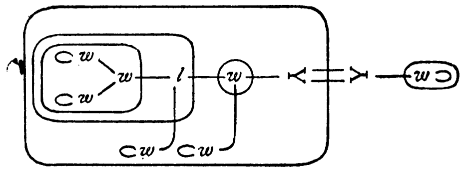

Fig. 26.

one thing is loved by all wise men." Fig. 27 means that every man is either wise or virtuous. Fig. 28 means that every man is both wise and virtuous.

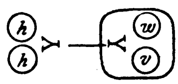

Fig. 27.

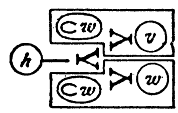

Fig. 28.

These explanations need not be carried further to show that we have here a perfectly efficient and highly analytical method of representing relations.

§ 7. *The Algebra of Dyadic Relatives.*—Although the primitive relatives are triadic, yet they may be represented with but little violence by means of dyadic relatives, provided we allow several attachments to one blank. For instance, A gives B to C, may be represented by saying A is the first party in the transaction D, B is subject of D, C is second party of D, D is a giving by the first party of the subject to the second party. Triadic relatives cannot conveniently be represented on one line of writing. These considerations led me to invent the algebra of dyadic relatives as a tolerably convenient substitute in many cases for the graphical method of representation. In place of the one "operation," or mode of conjunction of graphical method, there are in this algebra four operations.

For the purpose of this algebra, I entirely discard the idea that every compound relative consists of an antecedent and a consequent part. I consider the circle round the antecedent as a mere sign of negation, for which in the algebra I substitute an *obelus* over that antecedent. The line between antecedent and consequent, I treat as a sign of an "operation" by itself. It signifies that anything whatever being taken as correlate of the first written member,—antecedent or consequent,—and as first relate of the second written member, either the one or the other is to be accepted. Thus in place of the relative of Fig. 29 signifying that "taking anything whatever, M, either—is not a lover of M, or M is a benefactor of —," that is "—is a lover only of a benefactor of—," I write

$$\bar{l} \dagger b.$$

Or if it happens to be read the other way, putting a short mark over any letters to signify that relate and correlate are interchanged, I write the same thing

$$\breve{b} \dagger \breve{\bar{l}}.$$

This operation, which may, at need, be denoted by a dagger in print, to which I give a scorpion-tail curve in its cursive form, I call *relative addition.*

The relative "—stands to everything which is a benefactor of — in the relation of servant of every lover of his," shows,

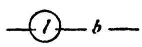

Fig. 29.

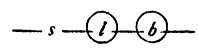

Fig. 30.

as written in Fig. 30, an unencircled bond between *s* and *l*. The junction of the *l* and the *b* may therefore be regarded as direct. Stating the relative so as to make this direct junction prominent, it is "—is servant of everything that is a lover of a benefactor of—." In the algebra, as far as already explained, "lover of a benefactor" would be written

$$\overline{\bar{l} \dagger \bar{b}}$$

that is, not a non-lover of every benefactor, or not a lover only of non-benefactors. This mode of junction, I call, in the algebra, the operation of *relative multiplication*, and write it

$$l b.$$

We have, then, the purely formal, or meaningless, equation

$$l b = \overline{\bar{l} \dagger \bar{b}}.$$

And in like manner, as a consequence of this,

$$l \dagger b = \overline{\bar{l} \bar{b}}.$$

That is to say, "To say that A is a lover of everything but benefactors of B," or "A is a non-lover only of benefactors of B," is the same as to say that A is not a non-lover of a non-benefactor of B.

To express in the algebra the relative of Fig. 31

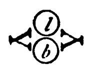

Fig. 31.

or "— is both a lover and a benefactor of —," I write

$$l \cdot b,$$

calling this "the operation of *non-relative multiplication.*" To express "— is either a lover or a benefactor of —," which might be written

$$\overline{\bar{l} \cdot \bar{b}},$$

I write

$$l \,+\!\!,\, b,$$

calling this the operation of *non-relative addition,* or more accurately, of *aggregation.* These last two operations belong to the Boolian algebra of non-relative logic. They are De Morgan's operations of composition and aggregation. Boole himself did not use the last, but in place of it an operation more properly termed addition which gives no interpretable result when the aggregants have any common aggregant. Mr. Venn still holds out for Boole's operation, and there are weighty considerations in its favor. In my opinion, the decision between the two operations should depend upon whether the quantified predicate is rejected (when aggregation should be used), or accepted (when Boole's strict addition should be used).

The use of these four operations necessitates continual resort to parentheses, brackets, and braces to show how far the different compound relatives extend. It also becomes desirable to have a "copula of inclusion," or the sign of "is exclusively (if anything)." For this purpose I have since 1870 employed the sign $-\!\!<$ (intended for an improved $\leqq$). It is easily made in the composing room from a dash followed by $<$, and in its cursive form is struck off in two rapid strokes, thus $-\!\!<$. Its meaning is exemplified in the formula

$$w -\!\!< v$$

"anybody who is wise (if any there be) is exclusively found among the virtuous." We also require in this algebra the signs of relatives of second intention

0, "— is inconsistent with —," ∞, "— is coexistent with —,"
T, "— is other than —," 1, "— is identical with."

The algebra has a moderate amount of power in skilful hands; but its great defect is the vast multitude of purely formal propositions which it brings along. The most significant of these are

$$s(l \dagger b) -\!\!< sl \dagger b$$

and

$$(l \dagger b)s -\!\!< l \dagger bs.$$

That is, whatever is a servant of something which is a lover of everything but benefactors is a servant-of-a-lover to everything but benefactors, etc.

Professor Schröder attaches, as it seems to me, too high a value to this algebra. That which is in his eyes the greatest recommendation of it is to me scarcely a merit, namely that it enables us to express in the outward guise of an equation propositions whose real meaning is much simpler than that of an equation.

§ 8. *General algebra of logic.*—Besides the algebra just described, I have invented another which seems to me much more valuable. It expresses with the utmost facility everything which can be expressed by a graph, and frequently much more clearly than the unabridged graphs described above. The method of using it in the solution of special problems has also been fully developed by me.

In this algebra every proposition consists of two parts, its quantifiers and its Boolian. The Boolian consists of a number of relatives united by a non-relative multiplication and aggregation. No relative operations are required (though they can be introduced if desired). Each elementary relative is represented by a letter on the line of writing with subjacent indices to denote the hecceities which fill its blanks. An obelus is drawn over such a relative to deny it.

To the left of the Boolian are written the quantifiers. Each of these is a Π or a Σ with one of the indices written subjacent to it, to signify that in the Boolian every object in the universe is to be imaged substituted successively for that index and the non-relative product (if the quantifier is Π) or the aggregate (if the quantifier is Σ) of the results taken. The order of the quantifiers is, of course, material. Thus

$$\Pi_i \Sigma_j l_{ij} = (l_{11} + l_{12} + l_{13} + \text{etc.}) \cdot (l_{21} + l_{22} + l_{23} + \text{etc.}) \cdot \text{etc.}$$

will mean anything loves something. But

$$\Sigma_j \Pi_i l_{ij} = l_{11} \cdot l_{21} \cdot l_{31} \cdot \text{etc.} + l_{12} \cdot l_{22} \cdot l_{32} \cdot \text{etc.} + l_{13} \cdot l_{23} \cdot l_{33} \cdot \text{etc.} + \text{etc.}$$

will mean something is loved by all things.

This algebra, which has but two operations, and those easily manageable, is, in my opinion, the most convenient apparatus for the study of difficult logical problems, although the graphical method is capable of such modification as to render it substantially as convenient on the average. Nor would I refuse to avail myself of the algebra of dyadic relatives in the simpler cases in which it is easily handled.

§ 9. *Method of Calculating with the General Algebra.*—My rules for working this algebra, the fruit of long experience with applying it to a great variety of genuine inquiries, have never been published. Nor can I here do more than state such as the beginner will be likely to require.

A number of premises being given, it is required to know the most important conclusions of a certain description which can be drawn from them. The first step will be to express the premises by means of the general algebra, taking care to use entirely different letters *as indices* in the different premises.

These premises are then to be copulated (or, in Whewell's phrase, colligated), i. e., non-relatively multiplied together, by multiplying their Boolians and writing before the product all the quantifiers. The relative order of the quantifiers of each premise must (in general) be undisturbed; but the relative order of quantifiers of different premises is arbitrary. The student ought to place Σ's as far to the left and Π's as far to the right as possible. Different arrangements of the quantifiers will lead to different conclusions from the premises. It sometimes happens that each of several arrangements leads to a conclusion which could not easily be reached from any other arrangement.

The premises, being so copulated, become one copulated premise. This copulated premise is next to be logically multiplied into itself any number of times, the indices being different in all the different factors. For there will be certain conclusions which I call conclusions of the first order, which can be drawn from the copulated premise without such involution, certain others, which I call inferences of the second order, which can be drawn from its square, etc. But after involution has been carried to a certain point, higher powers will only lead to inferences of subsidiary importance. The student will get a just idea of this matter by considering the rise and decline of interest in the theorems of any mathematical theory, such as geometry or the theory of numbers, as the fundamental hypotheses are applied more and more times in the demonstrations. The number of factors in the copulated premise, which embraces *all* the hypotheses that either theory assumes, is not great. Yet from this premise many thousand conclusions have already been drawn in the case of geometry and hundreds in the case of the theory of numbers. New conclusions are now coming in faster than ever before. From the nature of logic they can never be exhausted. But as time goes on the conclusions become more special and less important. It is true that mathematics, as a whole, does not become more special nor its late discoveries less important, because there is a growth of the hypotheses. Up to a certain degree, the importance of the conclusions increases with their "order." Thus, in geometry, there is nothing worth mention of the first order, and hardly of the second. But there is a great falling off in the importance of conclusions in the theories mentioned long before the fiftieth order has been reached.

This involution having been performed, the next step will be the identification (occasionally the diversification) of certain indices. The rule is, that any index quantified with a Π can be transmitted, throughout the Boolian, into any other index whose quantifier stands to the left of its own, which now becomes useless, since it refers to nothing in the Boolian. For example, in

$$\Sigma_i \, \Pi_j \, l_{ij}$$

which in the Algebra of Dyadic Relatives would be written ꝏ $(l$ ɟ $0)$, we can identify ɟ with i and write

$$\Sigma_i \, l_{ii}$$

which in the other algebra becomes ꝏ$(l \cdot 1)$ꝏ.

That done, the Boolian is to be manipulated according to any of the methods of non-relative Boolian algebra, and the conclusion is read off.

But it is only in the simplest cases that the above operations suffice. Relatives of second intention will often have to be introduced; and their peculiar properties must be attended to. Those of 0 and ꝏ are covered by the rules of non-relative Boolian algebra; but it is not so with 1 and T. We have, for example, to observe that

$$\Pi_i \, x_i +\!\!, \, y_i = \Pi_i \, \Pi_j \, x_i +\!\!, \, \mathrm{T}_{ij} +\!\!, \, y_i .$$

$$\Sigma_i \, x_i \cdot y_i = \Sigma_i \, \Sigma_j \, x_i \cdot 1_{ij} \cdot y_j .$$

Exceedingly important are the relatives signifying "— is a quality of —" and "— is a relation of — to —." It may be said that mathematical reasoning (which is the only deductive reasoning, if not absolutely, at least eminently) almost entirely turns on the consideration of abstractions as if they were objects. The protest of nominalism against such hypostatisation, although, if it knew how to formulate itself, it would be justified as against much of the empty disputation of the medieval Dunces, yet, as it was and is formulated, is simply a protest against the only kind of thinking that has ever advanced human culture. Nobody will work long with the logic of relatives,—unless he restricts the problems of his studies very much,—without seeing that this is true.

§ 10. *Schröder's Conception of Logical Problems.*.—Of my own labors in the logic of relatives since my last publication in 1884, I intend to give a slight hint in § 13. But I desire to give some idea of a part of the contents of Schröder's last volume. In doing so, I shall adhere to my own notation; for I cannot accept Professor Schröder's proposed innovations. I shall give my reasons in detail for this dissent in the *Bulletin of the American Mathematical Society*. I will here only indicate their general nature. I have no objection whatever to the creation of a new system of signs *ab ovo*, if anybody can propose such a system sufficiently recommending itself. But *that* Professor Schröder does not attempt. He wishes his notation to have the support of existing habits and conventions, while proposing a measure of reform in the present usage. For that he must obtain general consent. Now it seems to me quite certain that no such general agreement can be obtained without the strictest deference to the principle of priority. Without that, new notations can only lead to confusion thrice confounded. The experience of biologists in regard to the nomenclature of their genera and other groups shows that this is so. I believe that their experience shows that the only way to secure uniformity in regard to conventions of this sort, is to accept for each operation and relative the sign definitively recommended by the person who introduced that operation or relative into the Boolian algebra, unless there are the most *substantial* reasons for dissatisfaction with the meaning of the sign. Objections of lesser magnitude may justify slight modifications of signs; as I modify Jevons's ·|· to ✢, by uniting the two dots by a connecting line, and as I so far yield to Schröder's objections to using ∞ for the sign of whatever is, as to resort to the similarly shaped sign of Aries ♈ (especially as a notation of some power is obtained by using all the signs of the Zodiac in the same sense, as I shall show elsewhere). In my opinion, Professor Schröder alleges no sufficient reason for a single one of his innovations; and I further consider them as *positively* objectionable.

The volume consists of thirty-one long sections filling six hundred and fifty pages. I can, therefore, not attempt to do more than to exemplify its contents by specimens of the work selected as particularly interesting. Professor Schröder chiefly occupies himself with what he calls "solution-problems," in which it is required to deduce from a given proposition an *equation* of which one member consists in a certain relative determined in advance, while the other member shall not contain that relative. He rightly remarks that such problems often involve problems of elimination.

While I am not at all disposed to deny that the so-called "solution-problems," consisting in the ascertainment of the general forms of relatives which satisfy given conditions, are often of considerable importance, I cannot admit that the interest of logical study centres in them. I hold that it is usually much more to the purpose to express in the simplest way what a given premise discloses in regard to the constitution of a relative, whether that simplest expression is of the nature of an equation or not. Thus, one of Schröder's problems is, "Given $x \prec a$, required x,"—for instance, knowing that an opossum is a marsupial, give a description of the opossum. The so-called solution is $\sum_u = x u \cdot a$, or opossums embrace precisely what is common to marsupials and to some other class. In my judgment $x \prec a$ might with great propriety be called the solution of $\sum_u = x u \cdot a$. When the information contained in a proposition is not of the nature of an equation, why should we, by circumlocutions, insist upon expressing it in the form of an equation?

Professor Schröder attaches great importance to the generality of solutions. In my opinion, this is a mistake. It is not merely that he insists that solutions shall be *complete*, as for example when we require *every root* of a numerical equation, but further that they shall all be embraced under one algebraical expression. Upon that he insists and with that he is satisfied. Whether or not the "solution" is such as to exhibit anything of the real constitution of the relative which forms the first member of the equation he does not seem to care; at least, there is no apparent consideration of the question of how such a result can be secured.

Pure mathematics always selects for the subjects of its studies manifolds of perfect homogeneity; and thence it comes that for the problems which first present themselves general solutions are possible, which notwithstanding their generality, guide us at once to all the particular solutions. But even in pure mathematics the class of problems which are capable of solutions at once general and useful is an exceedingly limited one. All others have to be treated by subdivision of cases. That is what meets us everywhere in higher algebra. As for general solutions, they are for the most part trivial,—like the well-known and obvious test for a prime number that the continued product of all lesser numbers increased by 1 shall be divisible by that number. Only in those cases in which a general solution points the way to the particular solutions is it valuable; for it is only the particular solutions which picture to the mind the solution of a problem; and a form of words which fails to produce a definite picture in the mind is meaningless.

Professor Schröder endeavors to give the most general formula of a logical problem. It is in dealing with such very general and fundamental matters that the exact logician is most in danger of violating his own principles of exactitude. To seek a formula for all logical problems is to ask what it is, in general terms, that men inquire. To answer that question, my own logical proceeding would be to note that it asks what the essence of a question, in general, is. Now a question is a rational contrivance or device, and in order to understand any rational contrivance, experience shows that the best way is to begin by considering what circumstances of need prompted the contrivance, and then upon what general principle its action is designed to fill that need. Applying this general experience to the case before us, we remark that every question is prompted by some need,—that is, by some unsatisfactory condition of things, and that the object of asking the question is to fill that need by bringing reason to bear upon it and to do this by a hypnotically suggestive indication of that to which the mind has to apply itself. I do not know that I have ever, before this minute, considered the question what is the most general formulation of a problem in general; for I do not find much virtue in general formulæ. Nor do I think my answer to this question affords any particularly precious suggestion. But its ordinary character makes it all the better an illustration of the manner—or one of the manners—in which an exact logician may attack, off-hand, a suddenly sprung question. A question, I say, is an indication suggestive (in the hypnotic sense) of what has to be thought about in order to satisfy some more or less pressing want. Ideas like those of this statement, and not talk about φx, and "roots," and the like, must, in my opinion, form the staple of a logical analysis and useful description of a problem, in general. I am none the less a mathematical logician for that. If of two students of the theory of numbers one should insist upon considering numbers as expressed in a system of notation like the Arabic (though using now one number as base of the numeration, and now another), while the other student should maintain that all that was foreign to the theory of numbers, which ought not to consider upon what system the numbers with which it deals are expressed, those two students would, to my apprehension, occupy positions analogous to that of Schröder and mine in regard to this matter of the formulation of the problems of logic; and supposing the student who wished to consider the forms of expression of numbers were to accuse the other of being wanting in the spirit of an arithmetician, that charge would be unjust in quite the same way in which it would be unjust to charge me with deficiency in the mathematical spirit on account of my regarding the conceptions of "values," and "roots," and all that as very special ideas, which can only lumber up the field of consciousness with such hindrances as it is the very end and aim of that diagrammatic method of thinking that characterises the mathematician to get rid of.

But different questions are so very unlike that the only way to get much idea of the nature of a problem is to consider the different cases separately. There are in the first place questions about needs and their fulfillment which are not directly affected by the asking of the questions. A very good example is a chess problem. You have only to experiment in the imagination just as you would do on the board if it were permitted to touch the men, and if your experiments are intelligently conducted and are carried far enough, the solution required must be discovered. In other cases, the need to which the question relates is nothing but the intellectual need of having that question answered. It may happen that questions of this kind can likewise be answered by imaginary experimentation; but the more usual case requires real experimentation. The need is of one or other of two kinds. In the one class of cases we experience on several occasions to which our own deliberate action gave a common character, an excitation of one and the same novel idea or sensation, and the need is that a large number of propositions having the same novel consequent but different antecedents, should be replaced by one proposition which brings in the novel element, so that the others shall appear as mere consequences of every day facts with a single novel one. We may express this intellectual need in a brief phrase as the need of synthetising a multitude of subjects. It is the need of *generalisation*. In another class of cases, we find in some new thing, or new situation, a great number of characters, the same as would naturally present themselves as consequences of a hypothetical state of things, and the need is that the large number of novel propositions with one subject or antecedent should be replaced by a single novel proposition, namely that the new thing or new occasion belongs to the hypothetical class, from which all those other novelties shall follow as mere consequences of matters of course. This intellectual need, briefly stated, is the need of synthetising a multitude of predicates. It is the need of *theory*. Every problem, then, is either a problem of consequences, a problem of generalisation, or a problem of theory. This statement illustrates how special solutions are the only ones which directly mean anything or embody any knowledge; and general solutions are only useful when they happen to suggest what the special solutions will be.

Professor Schröder entertains very different ideas upon these matters. The general problem, according to him, is, "Given the proposition $Fx = 0$, required the 'value' of x_0," that is, an expression not containing x which can be equated to x. This 'value' must be the "general root," that is, it must, under one general description, cover every possible object which fulfils a given condition. This, by the way, is the simplest explanation of what Schröder means by a "solution-problem"; it is the problem to find that form of relative which necessarily fulfils a given condition and in which every relative that fulfils that condition can be expressed. Schröder shows that the solution of such a problem can be put into the form $\sum_u[x=fu]$, which means that a suitable logical function (f) of *any* relative, u, no matter what, will satisfy the condition $Fx=0$; and that nothing which is not equivalent to such a function will satisfy that condition. He further shows, what is very significant, that the solution may be required to satisfy the "adventitious condition" $fx=x$. This fact about the adventitious condition is all that prevents me from rating the value of the whole discussion as far from high.

Professor Schröder next produces what he calls "the rigorous solution" of the general question. This promises something very fine,—the rigorously correct resolution of everything that ever could (but for this knowledge) puzzle the human mind. It is true that it supposes that a particular relative has been found which shall satisfy the condition $Fx=0$. But that is seldom difficult to find. Either 0, or ∞, or some other trivial solution commonly offers itself. Supposing, then, that a be this particular solution, that is, that $Fa=0$, the "rigorous solution" is

$$x=fu=a\cdot\infty(Fu)\infty\boldsymbol{+}u\cdot(0\dagger\overline{Fu}\dagger 0).$$

That is, it is such a function of u that when u satisfies the condition $Fu=0$, $fu=u$; but when u does not satisfy this condition $fu=a$. Now $Fa=0$.

Since Professor Schröder carries his algebraicity so very far, and talks of "roots," "values," "solutions," etc., when, even in my opinion, with my bias towards algebra, such phrases are out of place, let us see how this "rigorous solution" would stand the climate of numerical algebra. What should we say of a man who professed to give rigorous general solutions of algebraic equations of every degree (a problem included, of course, under Professor Schröder's general problem)? Take the equation $x^5+Ax^4+Bx^3+Cx^2+Dx+E=0$. Multiplying by $x-a$ we get

$$x^6+(A-a)x^5+(B-aA)x^4+(C-aB)x^3+(D-aC)x^2+(E-aD)x-aE=0$$

The roots of this equation are precisely the same as those of the proposed quintic together with the additional root $x=a$. Hence, if we solve the sectic we thereby solve the quintic. Now, our

Schröderian solver would say, "There is a certain function, fu, every value of which, no matter what be the value of the variable, is a root of the sextic. And this function is formed by a direct operation. Namely, for all values of u which satisfy the equation

$$u^6+(A-a)u^5+(B-aA)u^4+(C-aB)u^3+(D-aC)u^2+(E-aD)u-aE=0$$

$fu=u$, while for all other values, $fu=a$.

Then, $x=fu$ is the expression of every root of the sextic and of nothing else. It is safe to say that Professor Schröder would pronounce a pretender to algebraical power who should talk in that fashion to be a proper subject for *surveillance* if not for confinement in an asylum. Yet he would only be applying Professor Schröder's "rigorous solution," neither more nor less. It is true that Schröder considers this solution as somewhat unsatisfactory; but he fails to state any principle according to which it should be so. Nor does he hold it too unsatisfactory to be frequently resorted to in the course of the volume. The *invention* of this solution exhibits in a high degree that very effective ingenuity which the *solution itself* so utterly lacks, owing to its resting on no correct conception of the nature of problems in general and of their solutions and of the meaning of a proposition.

§ 11. *Professor Schröder's Pentagrammatical Notation.*—Professor Schröder's greatest success in the logic of relatives, is due precisely to his having, in regard to certain questions, proceeded by the separation of cases, quite abandoning the glittering generalities of the algebra of dyadic relatives. As his greatest success, I reckon his solutions of "inverse row and column problems" in § 16, resting upon an investigation in § 15 of the relations of various compound relatives which end in 0, ∞, 1, and T. The investigations of § 15 might perfectly well have been carried through without any other instrument than the algebra of dyadic relatives. This course would have had certain advantages, such as that of exhibiting the principles on which the formulæ rest. But directness of proof would not have been of the number of those advantages; this is on the contrary decidedly with the notation invented and used by Professor Schröder. This notation may be called *pentagrammatic*, since it denotes a relative by a row of 5 characters. Imagine a list to be made of all the objects in the universe. Second, imagine a switchboard, consisting of a horizontal strip of brass for each object (these strips being fastened on a wall at a little distance one over another according to the order of the objects in the list) together with a vertical strip of brass for each object (these strips being fastened a little forward of the others, and being arranged in the same order), with holes at all the intersections, so that when a brass plug is inserted in any hole, the object corresponding to the horizontal brass strip can act in some way upon the object corresponding to the vertical brass strip. In order then, by means of this switchboard, to get an analogue of any dyadic relative, a lover of —," we insert plugs so that A and B, being any two objects, A can act on B, if and only if A is a lover of B. Now in Professor Schröder's pentagrammatic notation, the first of the five characters denoting any logical function of a primitive relative, a, refers to those horizontal strips, all whose holes are plugged in the representation of a (or, as we may say for short, "in a"), the second refers to those horizontal strips, each of which has in a every hole plugged but one. This one, not necessarily the same for all such strips, may be denoted by A. The third character refers to those horizontal strips which in a have several holes plugged, and several empty. The full holes (different, it may be, in the different horizontal strips) may be denoted by β. The fourth character refers to those horizontal strips which in a have, each of them, but one hole plugged, generally a different hole in each. This one plugged hole may be denoted by Γ. The fifth character will refer to those rows each of which in a has all its holes empty. Then, a will be denoted by $\infty\,\bar{\mathrm{A}}\,\beta\,\Gamma\,0$; and $\bar{a}$ by $0\,\mathrm{A}\,\bar{\beta}\,\bar{\Gamma}\,\infty$; for in $\bar{a}$, all the holes must be filled that are void in a, and *vice versa*. Consequently $\bar{a}\,\mathrm{T} = 0\,\bar{\mathrm{A}}\,\infty\;\infty\;\infty$. This shall be shown as soon as we have first examined the pentagrammatic symbol for a. This symbol divides a into four aggregants, viz:

$$a = (a \text{ ɟ } 0) + a \cdot [(a \text{ ɟ } 1) \cdot \bar{a}]\,\mathrm{T} + a \cdot a\,\mathrm{T} \cdot (\bar{a} \cdot \bar{a}\,\mathrm{T})\,\mathrm{T} + a \cdot (\bar{a} \text{ ɟ } 1)$$

In order to prove, by the algebra itself that this equation holds, we remark that $a = a \cdot b + a \cdot \bar{b}$, whatever b may be. For b, substitute

$(a \dagger 0)$. Then, $a \dagger 0 \prec a \dagger \mathrm{T}$; but $a \dagger \mathrm{T} = a$. Hence, $a \cdot b = a \dagger 0$. $a \cdot \bar{b} = a \cdot \bar{a}\,\infty = a \cdot \bar{a}(\mathrm{I} + \mathrm{T}) = a \cdot (\bar{a}\mathrm{I} + \bar{a}\mathrm{T})$. But $\bar{a}\mathrm{I} = \bar{a}$, and $a \cdot \bar{a} = 0$. Hence $a \cdot \bar{b} = a \cdot \bar{a}\mathrm{T}$. Thus $a = a \dagger 0 + a \cdot \bar{a}\mathrm{T}$. Now, in $\bar{a} = \bar{a} \cdot c + \bar{a} \cdot \bar{c}$, substitute for c, $a \dagger \mathrm{I}$. This gives $\bar{a} = (a \dagger \mathrm{I}) \cdot \bar{a} + \bar{a}\mathrm{T} \cdot \bar{a}$; and thus, $a = a \dagger 0 + a \cdot [(a \dagger \mathrm{I}) \cdot \bar{a}]\mathrm{T} + a \cdot (\bar{a}\mathrm{T} \cdot \bar{a})\mathrm{T}$. Finally, $a = a \cdot a\mathrm{T} + a \cdot (\bar{a} \dagger \mathrm{I})$. But $a \cdot (\bar{a} \dagger \mathrm{I}) = a \cdot (\bar{a} \dagger \mathrm{I}) \cdot (\bar{a}\mathrm{T} \cdot \bar{a})\mathrm{T} + a \cdot (\bar{a} \dagger \mathrm{I}) \cdot \{[(a \dagger \mathrm{I}) + a] \dagger \mathrm{I}\}$.

And

$$
\begin{aligned}
a \cdot (\bar{a} \dagger \mathrm{I}) \cdot \{[(a \dagger \mathrm{I}) + a] \dagger \mathrm{I}\} &= a \cdot \{\bar{a} \cdot [(a \dagger \mathrm{I}) + a] \dagger \mathrm{I}\} && \text{(by distribution)} \\
&= a \cdot [\bar{a} \cdot (a \dagger \mathrm{I}) \dagger \mathrm{I}] && (\text{since } \bar{a} \cdot a = 0) \\
&= a \cdot (\bar{a} \dagger \mathrm{I}) \cdot (a \dagger \mathrm{I} \dagger \mathrm{I}) && \text{(by distribution)} \\
&= a \cdot (\bar{a} \dagger \mathrm{I}) \cdot (a \dagger 0) && \text{(if more than 2 things exist)} \\
&= a \cdot (\bar{a} \dagger \mathrm{I}) \cdot (a \dagger \mathrm{I} \cdot \mathrm{T}) && (\text{since } 0 = \mathrm{I} \cdot \mathrm{T}) \\
&= a \cdot (\bar{a} \dagger \mathrm{I}) \cdot (a \dagger \mathrm{I}) \cdot (a \dagger \mathrm{T}) && \text{(by distribution)} \\
&= a \cdot (\bar{a} \dagger \mathrm{I}) \cdot (a \dagger \mathrm{I}) && (\text{since } a \dagger \mathrm{T} = a) \\
&= a \cdot (\bar{a} \cdot a \dagger \mathrm{I}) && \text{(by distribution)} \\
&= a \cdot (0 \dagger \mathrm{I}) && (\text{since } \bar{a} \cdot a = 0) \\
&= a \cdot 0 && \text{(if more than 1 object exists)} \\
&= 0.
\end{aligned}
$$

So that $a \cdot (\bar{a} \dagger \mathrm{I}) = a \cdot (\bar{a} \dagger \mathrm{I}) \cdot (\bar{a}\mathrm{T} \cdot \bar{a})\mathrm{T}$ and thus

$$a = a \dagger 0 + a \cdot [(a \dagger \mathrm{I}) \cdot \bar{a}]\mathrm{T} + a \cdot a\mathrm{T}(\bar{a}\mathrm{T} \cdot \bar{a})\mathrm{T} + a \cdot (\bar{a} \dagger \mathrm{I}).$$

This is the meaning of the symbol $\infty \bar{\mathrm{A}} \beta \Gamma 0$.

We, now, at length, return, as promised to the examination of $\bar{a}\mathrm{T}$. First, $a \dagger 0 \prec \overline{\bar{a}\mathrm{T}} \dagger 0$. For $\overline{\bar{a}\mathrm{T}} = a \dagger \mathrm{I}$ and $a \dagger \mathrm{I} \dagger 0 = a \dagger (\mathrm{I} \dagger 0) = a \dagger 0$. Hence the first character in the pentagrammatic symbol for $\bar{a}\mathrm{T}$ must be 0. Second $a \cdot [(a \dagger \mathrm{I}) \cdot \bar{a}]\mathrm{T} \prec \bar{a}\mathrm{T} \cdot [(\bar{a}\mathrm{T} \dagger \mathrm{I}) \cdot \overline{\bar{a}\mathrm{T}}]\mathrm{T}$. For it is plain that $a \cdot [(a \dagger \mathrm{I}) \cdot \bar{a}]\mathrm{T} \prec [(a \dagger \mathrm{I}) \cdot \bar{a}]\mathrm{T} \prec \bar{a}\mathrm{T}$. Also $\bar{a} \prec \bar{a}\,\infty \prec \bar{a}(\mathrm{T} \dagger \mathrm{I}) \prec \bar{a}\mathrm{T} \dagger \mathrm{I}$. Hence $[(a \dagger \mathrm{I}) \cdot \bar{a}]\mathrm{T} \prec [(a \dagger \mathrm{I}) \cdot (\bar{a}\mathrm{T} \dagger \mathrm{I})]\mathrm{T}$. But $a \dagger \mathrm{I} = \overline{\bar{a}\mathrm{T}}$. Hence, $a \cdot [(a \dagger \mathrm{I}) \cdot \bar{a}]\mathrm{T} \prec \bar{a}\mathrm{T} \cdot [(\bar{a}\mathrm{T} \dagger \mathrm{I}) \cdot \overline{\bar{a}\mathrm{T}}]\mathrm{T}$. Hence, the second character in the pentagrammatic sign for $\bar{a}\mathrm{T}$, is the same as that of a. Thirdly $a \cdot a\mathrm{T} \cdot (\bar{a}\mathrm{T} \cdot \bar{a})\mathrm{T} \prec \bar{a}\mathrm{T} \dagger 0$. For $\bar{a} \prec \bar{a}\mathrm{I} \prec \bar{a}(\mathrm{T} \dagger \mathrm{I}) \prec \bar{a}\mathrm{T} \dagger \mathrm{I}$. Hence $(\bar{a} \cdot \bar{a}\mathrm{T})\mathrm{T} \prec [(\bar{a}\mathrm{T} \dagger \mathrm{I}) \cdot (\bar{a}\mathrm{T} \dagger \mathrm{T})]\mathrm{T} \prec (\bar{a}\mathrm{T} \dagger \mathrm{I} \cdot \mathrm{T})\mathrm{T} \prec (\bar{a}\mathrm{T} \dagger 0)\mathrm{T} \prec \bar{a}\mathrm{T} \dagger 0\mathrm{T} \prec \bar{a}\mathrm{T} \dagger 0$. Consequently, the third character of the pentagrammatic symbol of $\bar{a}\mathrm{T}$ must be ∞.

Fourthly, $a\cdot(\bar{a}\dagger\mathrm{I})\prec\bar{a}\mathrm{T}\dagger 0$. For we have just seen that $\bar{a}\prec\bar{a}\mathrm{T}\dagger\mathrm{I}$. Hence $\bar{a}\dagger\mathrm{I}\prec\bar{a}\mathrm{T}\dagger\mathrm{I}\dagger\mathrm{I}$. But $\mathrm{I}\dagger\mathrm{I}=0$ if there is more than one object in the universe. Hence $\bar{a}\dagger\mathrm{I}\prec\bar{a}\mathrm{T}\dagger 0$. Consequently, the fourth character of the pentagrammatic formula for $\bar{a}\mathrm{T}$ is ∞. Finally, $\bar{a}\dagger 0\prec\bar{a}\mathrm{T}\dagger 0$. For $\bar{a}\dagger 0\prec\bar{a}\dagger 0\dagger 0\prec\bar{a}\dagger\mathrm{I}\cdot\mathrm{T}\dagger 0\prec(\bar{a}\dagger\mathrm{I})\cdot(\bar{a}\dagger\mathrm{T})\dagger 0\prec\bar{a}\dagger\mathrm{I}\dagger 0\prec\bar{a}\mathrm{T}\dagger 0$. Hence the fifth character of the pentagram of $\bar{a}\mathrm{T}$ is ∞. In fine, that pentagram is $0\bar{\mathrm{A}}\infty\infty\infty$. Professor Schröder obtains this result more directly by means of a special calculus of the pentagrammatic notation. In that way, he obtains, in § 15, a vast number of formulæ, which in § 16 are applied in the first place with great success to the solution of such problems as this: Required a form of relation in which everything stands to something but nothing to everything. The author finds instantaneously that every relative signifying such a relation must be reducible to the form $\bar{u}\infty\cdot u + \mathrm{I}\cdot(u\dagger 0 + \bar{u}\dagger 0)$. In fact, the first term of this expression $\bar{u}\infty\cdot u$, for which $\bar{u}\infty\cdot u\infty$ might as well be written, embraces all the relatives in question. For let $\bar{u}$ be any such relative. Then, $u=\bar{u}\infty\cdot u$. The second term is added, curiously enough, merely to *exclude other relations.* For if u is such a relative that something is u to everything or to nothing, then that something would be in the relation $\bar{u}\infty\cdot u$ to nothing. To give it a correlate the second term is added; and since all the relatives are already included, it matters not what that correlate be, so long as the second term does not exclude any of the required relatives which are included under the first term. Let v be any relative of the kind required, then $v\cdot(u\dagger 0 + \bar{u}\dagger 0)$ will answer for the second term. If we had no letter expressing a relation known to be of the required kind, the problem would be impossible. Fortunately, both I and T are of that kind. Of course, the negative of such a relative is itself such a relative; so that

$$(u\dagger 0\,\bar{u}\dagger 0)\cdot(v + u\infty\cdot\bar{u}\infty)$$

would be an equivalent form, equally with

$$(u\dagger 0 + \bar{u}\dagger 0)\cdot v + u\infty\cdot\bar{u}\infty.$$

§ 16 concludes with some examples of eliminations of great apparent complexity. In the first of these we have given $x=$

$(\bar{u}\,\text{ɟ}\,\mathsf{1})\text{ꝏ}\,✛\,u$; and it is required to eliminate u. We have, however, instantly $u \mathrel{⋖} x$

$$(\bar{u}\,\text{ɟ}\,\mathsf{1})\text{ꝏ} \mathrel{⋖} x$$

Whence, immediately,

$$(\bar{x}\,\text{ɟ}\,\mathsf{1})\text{ꝏ} \mathrel{⋖} x,$$

or

$$\text{ꝏ} \mathrel{⋖} (x \cdot x\mathsf{T})\text{ꝏ}.$$

The next example, the most complicated, requires u to be eliminated from the equation

$$x = \bar{u}\,\text{ɟ}\,0\,✛\,(u\,\text{ɟ}\,\mathsf{1})\text{ꝏ}\cdot\bar{u}\mathsf{T}\,✛\,(u\,\text{ɟ}\,\mathsf{1})\cdot\bar{u}\,✛\,(\bar{u}\,\text{ɟ}\,\mathsf{1})\cdot u\,✛\,(u\mathsf{T}\cdot\bar{u}\mathsf{T}\,\text{ɟ}\,0)\cdot\bar{u},$$

He performs the elimination by means of the pentagrammatic notation very easily as follows: Putting $u = \text{ꝏ}\,\bar{\mathrm{A}}\beta\Gamma 0$

$$\begin{array}{rl} \bar{u}\,\text{ɟ}\,0 & = 0\,0\,0\,0\,\text{ꝏ} \\ (u\,\text{ɟ}\,\mathsf{1})\text{ꝏ}\cdot\bar{u}\mathsf{T} & = 0\,\bar{\mathrm{A}}\,0\,0\,0 \\ (u\,\text{ɟ}\,\mathsf{1})\cdot\bar{u} & = 0\,\mathrm{A}\,0\,0\,0 \\ (\bar{u}\,\text{ɟ}\,\mathsf{1})\cdot u & = 0\,0\,0\,\Gamma\,0 \\ (u\mathsf{T}\cdot\bar{u}\mathsf{T}\,\text{ɟ}\,0)\cdot\bar{u} & = 0\,0\,\bar{\beta}\,0\,0 \\ \hline \text{sum} & 0\,\text{ꝏ}\,\bar{\beta}\,\Gamma\,\text{ꝏ} \end{array}$$

Thus, x is of the form $\text{ꝏ}\,\text{-}\beta\Gamma 0$, which has been found in former problems to imply $x\,\text{ɟ}\,\mathsf{1} \mathrel{⋖} x$.

Without the pentagrammatical notation this elimination would prove troublesome, although with that as a guide it could easily be obtained by the algebra alone.

§ 12. *Professor Schröder's Iconic Solution of* $x \mathrel{⋖} \varphi x$.

Another valuable result obtained by Professor Schröder is the solutions of the problem

$$x \mathrel{⋖} \varphi x.$$

Namely, he shows that

$$x = f^{\infty} u$$

where

$$fu = u \cdot \varphi u$$

[Of course, by contraposition, this gives for the solution of $\varphi x \mathrel{⋖} x$ $x = f^{\infty} u$ where $fu = u\,✛\,\varphi u$.] The correctness of this solution will appear upon a moment's reflexion; and nearly all the useful solutions in the volume are cases under this.

It happens very frequently that the iteration of the functional operation is unnecessary, because it has no effect.

Suppose, for example, that we desire the general form of a "transitive" relative, that is, such a one, x, that

$$x x \prec x.$$

In this case, since $1 \prec l \dagger \breve{l}$ whatever l may be, we have

$$x \prec x 1 \prec x (x \dagger \breve{x}) \prec x x \dagger \breve{x} \prec x \dagger \breve{x},$$

or

$$x \prec x \dagger \breve{x}$$

If, then,

$$f u = u \cdot (u \dagger \breve{u}),$$

we have

$$x = f^{\infty} u.$$

Here,

$$f u \prec u;$$

so that

$$f^{\infty} u \prec f u.$$

Also,

$$f^2 u = f u \cdot (f u \dagger \breve{fu}) = u \cdot (u \dagger \breve{u}) \cdot [u \cdot (u \dagger \breve{u}) \dagger (\breve{u} + u \breve{u})]$$
$$= u \cdot (u \dagger \breve{u}) \cdot [u f (1 + u) \breve{u}] \cdot [u \dagger \breve{u} \dagger (1 + u) \breve{u}].$$

Now

$$f u = u \cdot (u \dagger \breve{u}) = u \cdot (u \dagger \breve{u}) \cdot (u \dagger \breve{u}) \cdot (u \dagger \breve{u}) = u \cdot (u \dagger \breve{u}) \cdot (u \dagger 1 \breve{u}) \cdot (u \dagger 1 \breve{u})$$
$$\prec u \cdot (u \dagger \breve{u}) \cdot [u \dagger (1 + u) \breve{u}] \cdot [u \dagger (\breve{u} \dagger u) \breve{u}] \prec$$
$$\prec u \cdot (u \dagger \breve{u}) \cdot [u \dagger (1 + u) \breve{u}] \cdot (u \dagger \breve{u} \dagger u \breve{u})$$
$$\prec u \cdot (u \dagger \breve{u}) \cdot [u \dagger (1 + u) \breve{u}] \cdot [u \dagger \breve{u} \dagger (1 + u) \breve{u}] \prec f^2 u.$$

Thus $f u = f^{\infty} u$; and

$$x = \sum_u u \cdot (u \dagger \breve{u})$$

This is a truly iconic result; that is, it shows us what the constitution of a transitive relative really is. It shows us that transitiveness always depends upon inclusion; for to say that A is $l \dagger \breve{l}$ of B is to say that the things loved by B are included among those loved by A. The factor $u \dagger \breve{u}$ is transitive by itself; for

$$(u \dagger \breve{u})(u \dagger \breve{u}) \prec u \dagger \breve{u} u \dagger \breve{u} \prec u \dagger \top \dagger \breve{u} \prec u \dagger \breve{u}.$$

The effect of the other factor, u, of the form for the general transitive is merely in certain cases to exclude universal identity, and thus to extend the class of relatives represented by $u \curlyvee \breve{u}$ so as to include those of which it is not true that $1 \prec x$. Here we have an instance of restriction having the effect of extension, that is, restriction of special relatives extends the class of relatives represented. This does not take place in all cases, but only where certain relatives can be represented in more than one way.

Indicating, for a moment, the copula by a dash, the typical and fundamental syllogism is

$$A - B \quad B - C$$
$$\therefore A - C.$$

That is to say, the principle of this syllogism enters into every syllogism. But to say that this is a valid syllogism is merely to say that the copula expresses a transitive relation. Hence, when we now find that transitiveness always depends upon inclusion, the initial analysis by which the copula of inclusion was taken as the general one is fully confirmed. For the chief end of formal logic is the representation of the syllogism.

§ 13. *Introduction to the Logic of Quantity.*—The great importance of the idea of quantity in demonstrative reasoning seems to me not yet sufficiently explained. It appears, however, to be connected with the circumstance that the relations of being greater than and of being at least as great as are transitive relations. Still, a satisfactory evolutionary logic of mathematics remains a desideratum. I intend to take up that problem in a future paper. Meantime the development of projective geometry and of geometrical topics has shown that there are at least two large mathematical theories of continuity into which the idea of continuous *quantity*, in the usual sense of that word, does not enter at all. For projective geometry Schubert has developed an algebraical calculus which has a most remarkable affinity to the Boolian algebra of logic. It is, however, imperfect, in that it only gives imaginary points, rays, and planes, without deciding whether they are real or not. This defect cannot be remedied until topology—or, as I prefer to call it, mathematical topics—has been further developed and its logic accurately analysed. To do this ought to be one of the first tasks of exact logicians. But before that can be accomplished, a perfectly satisfactory logical account of the conception of continuity is required. This involves the definition of a certain kind of infinity; and in order to make that quite clear, it is requisite to begin by developing the logical doctrine of infinite multitude. This doctrine still remains, after the works of Cantor, Dedekind, and others, in an inchoate condition. For example, such a question remains unanswered as the following: Is it, or is it not, logically possible for two collections to be so multitudinous that neither can be put into a one-to-one correspondence with a part or the whole of the other? To resolve this problem demands, not a mere *application* of logic, but a further *development* of the conception of logical possibility.

I formerly defined the possible as that which in a given state of information (real or feigned) we do not know not to be true. But this definition to-day seems to me only a twisted phrase which, by means of two negatives, conceals an anacoluthon. We know in advance of experience that certain things are not true, because we see they are impossible. Thus, if a chemist tests the contents of a hundred bottles for fluorine, and finds it present in the majority, and if another chemist tests them for oxygen and finds it in the majority, and if each of them reports his result to me, it will be useless for them to come to me together and say that they know infallibly that fluorine and oxygen cannot be present in the same bottle; for I see that such infallibility is *impossible*. I know it is not true, because I satisfy myself that there is no room for it even in that ideal world of which the real world is but a fragment. I need no sensible experimentation, because ideal experimentation establishes a much broader answer to the question than sensible experimentation could give. It has come about through the agencies of development that man is endowed with intelligence of such a nature that he can by ideal experiments ascertain that in a certain universe of logical possibility certain combinations occur while others do not occur. Of those which occur in the ideal world some do and some do not occur in the real world; but all that occur in the real world occur also in the ideal world. For the real world is the world of sensible experience, and it is a part of the process of sensible experience to locate its facts in the world of ideas. This is what I mean by saying that the sensible world is but a fragment of the ideal world. In respect to the ideal world we are virtually omniscient; that is to say, there is nothing but lack of time, of perseverance, and of activity of mind to prevent our making the requisite experiments to ascertain positively whether a given combination occurs or not. Thus, every proposition about the ideal world can be ascertained to be either true or false. A description of thing which occurs in that world is *possible, in the substantive logical sense.* Very many writers assert that everything is logically possible which involves no contradiction. Let us call that sort of logical possibility, *essential,* or *formal,* logical possibility. It is not the only logical possibility; for in this sense, two propositions contradictory of one another may both be severally possible, although their combination is not possible. But in the *substantive* sense, the contradictory of a possible proposition is impossible, because we are virtually omniscient in regard to the ideal world. For example, there is no contradiction in supposing that only four, or any other number, of independent atoms exist. But it is made clear to us by ideal experimentation, that five atoms are to be found in the ideal world. Whether all five are to be found in the sensible world or not, to say that there are only four in the ideal world is a proposition absolutely to be rejected, notwithstanding its involving no contradiction.

It would be a great mistake to suppose that ideal experimentation can be performed without danger of error; but by the exercise of care and industry this danger may be reduced indefinitely. In sensible experimentation, no care can always avoid error. The results of induction from sensible experimentation are to afford some ratio of frequency with which a given consequence follows given conditions in the existing order of experience. In induction from ideal experimentation, no particular order of experience is forced upon us; and consequently no such numerical ratio is deducible. We are confined to a dichotomy: the result either is that some description of thing occurs or that it does not occur. For example, we cannot say that one number in every three is divisible by three and one in every five is divisible by five. This is, indeed, so if we choose to arrange the numbers in the order of counting; but if we arrange them with reference to their prime factors, just as many are divisible by one prime as by another. I mean, for instance, when they are arranged as follows:

1, 2, 4, 8, etc.	5, 10, 20, 40, etc.	7, 14, 28, 56, etc.	35, 70, etc.
3, 6, 12, 24, etc.	15, 30, 60, 120, etc.	21, 42, 84, 168, etc.	105, 210, etc.
9, 18, 36, 72, etc.	45, 90, 180, 360, etc.	etc.	etc.
27, 54, 108, 16, etc.	135, 270, 540, 1080, etc.		
etc.	etc.		

Thus, dichotomy rules the ideal world. Plato, therefore, for whom that world alone was real, showed that insight into concepts but dimly apprehended that has always characterised philosophers of the first order, in holding dichotomy to be the only truthful mode of division. Lofty moral sense consists in regarding, not indeed *the*, but yet *an*, ideal world as in some sense the only real one; and hence it is that stern moralists are always inclined to dual distinctions.

Ideal experimentation has one or other of two forms of results. It either proves that $\Sigma_i m_i$, a particular proposition true of the ideal world, and going on, finds $\Sigma_j \bar{m}_j$ also true; that is, that m and $\bar{m}$ are both possible, or it succeeds in its induction and shows the universal proposition $\Pi_i m_i$ to be true of the ideal world; that is that $\bar{m}$ is *necessary* and m *impossible.*

Every result of an ideal induction clothes itself, in our modes of thinking, in the dress of a *contradiction.* It is an anacoluthon to say that a proposition is impossible *because* it is selfcontradictory. It rather is thought so as to appear selfcontradictory, because the ideal induction has shown it to be impossible. But the result is that in the absence of any interfering contradiction every particular proposition is possible in the substantive logical sense, and its contradictory universal proposition is impossible. But where contradiction interferes this is reversed.

In former publications I have given the appellation of *universal* or *particular* to a proposition according as its *first* quantifier is Π or Σ. But the study of substantive logical possibility has led me to substitute the appellations *negative* and *affirmative* in this sense, and to call a proposition *universal* or *particular* according as its *last* quantifier is Π or Σ. For letting l be any relative, one or other of the two propositions

$$\Pi_i \Sigma_j l_{ij} \qquad \Sigma_i \Pi_j \bar{l}_{ij}$$

and one or other of the two propositions

$$\Pi_j \Sigma_i \bar{l}_{ij} \qquad \Sigma_j \Pi_i l_{ij}$$

are true, while the other one of each pair is false. Now, in the absence of any peculiar property of the special relative l, the two similar forms $\Sigma_i \Pi_j \bar{l}_{ij}$ and $\Sigma_j \Pi_i l_{ij}$ must be equally possible in the substantive logical sense. But these two propositions cannot both be true. Hence, both must be false in the ideal world, in the absence of any constraining contradiction. Accordingly, these ought to be regarded as universal propositions, and their contradictions, $\Pi_i \Sigma_j l_{ij}$ and $\Pi_j \Sigma_i \bar{l}_{ij}$, as particular propositions.

There are two opposite points of view, each having its logical value, from one of which, of two quantifiers of the same proposition, the preceding is more important than the following, while from the other point of view the reverse is the case. Accordingly, we may say that an affirmative proposition is particular in a secondary way, and that a particular proposition is affirmative in a secondary way.

If an index is not quantified at all, the proposition is, with reference to that index, *singular*. To ascertain whether or not such a proposition is true of the ideal world, it must be shown to depend upon some universal or particular proposition.

If some of the quantifiers refer not to hecceities, having in themselves no general characters except the logical characters of identity, diversity, etc., but refer to *characters*, whether non-relative or relative, these alone are to be considered in determining the "quantity" of an ideal proposition as universal or particular. For anything whatever is true of *some* character, unless that proposition be downright absurd; while nothing is true of *all* characters except what is formally necessary. Consider, for example, a dyadic relation. This is nothing but an aggregation of pairs. Now any two hecceities may in either order form a pair; and any aggregate whatever of such pairs will form *some* dyadic relation. Hence, we may totally disregard the manner in which the hecceities are connected in determining the possibility of a hypothesis about *some* dyadic relation.

Characters have themselves characters, such as importance, obviousness, complexity, and the like. If some of the quantified indices denote such characters of characters, they will, in reference to a purely ideal world be paramount in determining the quantity of the proposition as universal or particular.

All quantitative comparison depends upon a *correspondence*. A correspondence is a relation which every subject[1] of one collection bears to a subject of another collection, to which no other is in the same relation. That is to say, the relative "corresponds to" has

$$\sum_u u\cdot(1\,ɟ\,\bar{u})$$

not merely as its *form*, but as its *definition*. This relative is transitive; for its relative product into itself is

$$[\sum_u u\cdot(1\,ɟ\,\bar{u})][\sum_v v\cdot(1\,ɟ\,\bar{v})] \prec \sum_u\sum_v uv\cdot(1\,ɟ\,\bar{u})(1\,ɟ\,\bar{v})$$
$$\prec \sum_u\sum_v uv\cdot(1\,ɟ\,\bar{u}\,ɟ\,\bar{v}) \prec \sum_u\sum_v uv\cdot(1\,ɟ\,\overline{uv}) \prec \sum_w w\cdot(1\,ɟ\,\bar{w})$$

But it is to be observed that if the P's, the Q's, and the R's are three collections, it does not follow because every P corresponds to an R, and every Q corresponds to an R that every object of the aggregate collection $P + Q$ corresponds to an R. The *dictum de omni* in external appearance fails here. For P may be $[u\cdot(1\,ɟ\,\bar{u})]R$ and Q may be $[v\cdot(1\,ɟ\,\bar{v})]R$; but the aggregate of these is not $[(u+v)\cdot(1\,ɟ\,\overline{u+v})]R$, which equals $[(u+v)\cdot(1\,ɟ\,\bar{u})\cdot(1\,ɟ\,\bar{v})]R$. The aggregate of the two first is $\{(u\,ɟ\,v)\cdot[v\cdot(1\,ɟ\,\bar{v}) + 1\,ɟ\,\bar{u}]\cdot[u\cdot(1\,ɟ\,\bar{u}) + 1\,ɟ\,\bar{v}]\}R$, which is obviously too broad to be necessarily included under the other expression. Correspondence is, therefore, not a relation between the subjects of one collection and those of another, but between the collections themselves. Let $q_{\alpha i}$ mean that i is a subject of the collection, α, and let $r_{\beta jk}$ mean that j stands in the relation β to k. Then, to say that the collection P corresponds to the collection Q, or, as it is sometimes expressed, that "*for* every

[1] I prefer to speak of a member of a collection as a *subject* of it rather than as an *object* of it; for in this way I bring to mind the fact that the collection is virtually a quality or class-character.

subject of Q there is a subject of P," is to make the assertion expressed by

$$\Sigma_\beta \Pi_i \Sigma_j \Pi_k \bar{q}_{Pi} + r_{\beta ij} \cdot (1_{ik} + \bar{r}_{\beta kj}) \cdot q_{Qj}.$$

In the algebra of dual relatives this may be written

$$\Sigma_\beta \mathrm{P} \prec \bar{q} \dagger [r_\beta \cdot (1 \dagger \bar{r}_\beta)] \breve{q} \mathrm{Q}.$$

The transitivity is evident; for

$$\Sigma_\beta \Sigma_\gamma \bar{q} \dagger [r_\beta \cdot (1 \dagger \bar{r}_\beta)] \breve{q} \{ \bar{q} \dagger [r_\gamma \cdot (1 \dagger \bar{r}_\gamma)] \breve{q} \}$$
$$\prec \Sigma_\beta \Sigma_\gamma \bar{q} \dagger [r_\beta \cdot (1 \dagger \bar{r}_\beta)] \{ \breve{q} \bar{q} \dagger [r_\gamma \cdot (1 \dagger \bar{r}_\gamma)] \breve{q} \}$$
$$\prec \Sigma_\beta \Sigma_\gamma \bar{q} \dagger [r_\beta \cdot (1 \dagger \bar{r}_\beta)] \{ \mathrm{T} \dagger [r_\gamma \cdot (1 \dagger \bar{r}_\gamma)] \breve{q} \}$$
$$\prec \Sigma_\beta \Sigma_\gamma \bar{q} \dagger [r_\beta \cdot (1 \dagger \bar{r}_\beta)] [r_\gamma \cdot (1 \dagger \bar{r}_\gamma)] \breve{q}$$
$$\prec \Sigma_\beta \Sigma_\gamma \bar{q} \dagger [r_\beta r_\gamma \cdot (1 \dagger \bar{r}_\beta \dagger \bar{r}_\gamma)] \breve{q}$$
$$\prec \Sigma_\delta \bar{q} \dagger [r_\delta \cdot (1 \dagger \bar{r}_\delta)] \breve{q}.*$$

Not only is the relative of correspondence transitive, but it also possesses what may be called *antithetic transitivity*. Namely, if c be the relative, not only is $cc \prec c$ but also $c \prec c \dagger c$. To demonstrate this very important proposition is, however, far from easy. The quantifiers of the assertion that for every subject of one character there is a subject of another are $\Sigma_\beta \Pi_i \Sigma_j \Pi_k$. Hence, the proposition is particular and will be true in the ideal world, except in case a positive contradiction is involved.

Let us see how such contradiction can arise. The assertion that for every subject of P there is a subject of Q is

$$\Sigma_\beta \Pi_i \Sigma_j \Pi_k \bar{q}_{Pi} + r_{\beta ij} \cdot (1_{ik} + \bar{r}_{\beta ki}) \cdot q_{Qj}.$$

This cannot vanish if the first aggregant term does not vanish, that is, if $\Pi_i q_{Pi}$ or there is no subject of P. It cannot vanish if everything is a subject of Q. For in that case, the last factor of the latter aggregant disappears, and substituting 1 for r_β the second aggregant becomes ∞. The expression cannot vanish if every subject of P is a subject of Q. For when 1 is substituted for r_β, we get

$$\Pi_i \bar{q}_{Pi} + q_{Qi}.$$

If P has but a single individual subject and Q has a subject, for every P there is a Q. For in this case we have only to take for β

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*It must be remembered that to a person familiar with the algebra all such series of steps become evident at first glance.

the relation of the subject of P to any one of the subjects of Q. But if P has more than one subject, and Q has but one, the expression above vanishes. For let 1 and 2 be the two subjects of P. Substituting 1 for i, we get

$$\Pi_k\, r_{\beta 1 j} \cdot (1_{1k} + \bar{r}_{\beta kj}) \cdot q_{Qj}$$

Substituting 2 for i we get

$$\Pi_k\, r_{\beta 2 j} \cdot (1_{2k} + \bar{r}_{\beta kj}) \cdot q_{Qj}$$

Multiplying these

$$\Pi_k\, \Pi_k\, r_{\beta 1 j} \cdot r_{\beta 2 j} \cdot (1_{1k} + \bar{r}_{\beta kj}) \cdot (1_{2k'} + \bar{r}_{\beta kj}) \cdot q_{Qi}$$

Substituting 2 for k and 1 for k', this gives

$$r_{\beta 1 j} \cdot r_{\beta 2 j} \cdot \bar{r}_{\beta 2 j} \cdot {}_{\beta 1 j} \cdot Q_{qi}$$

which involves two contradictions.

It is to be remarked that although if every subject of P is a subject of Q, then for every subject of P there is a subject of Q, yet it does not follow that if the subjects of P are a part only of the subjects of Q, that there is then not a subject of P for every subject of Q. For example, numbering 2, 4, 6, etc., as the 1st, 2nd, 3rd, etc., of the even numbers, there is an even number for every whole number, although the even numbers form but a part of the whole numbers.

It is now requisite, in order to prove that $c \prec c \mathfrak{J} c$, to draw three propositions from the doctrine of substantive logical possibility. The first is that given any relation, there is a possible relation which differs from the given relation only in excluding any of the pairs we may choose to exclude. Suppose, for instance, that for every subject of P there is a subject of Q, that is that

$$\Sigma_\beta\ \breve{q}\,\mathrm{P} \prec [r_\beta \cdot (1 \mathfrak{J} \bar{r}_\beta)]\breve{q}\,\mathrm{Q}.$$

The factor $(1 \mathfrak{J} \bar{r}_\beta)$ here has the effect of allowing each correlate but one relate. Each relate is, however, allowed any number of correlates. If we exclude all but one of these, the one retained being, if possible, a subject of Q, we have a possible relation, β', such that

$$\Sigma_{\beta'}\ \breve{q}\,\mathrm{P} \prec [r_{\beta'} \cdot (1 \mathfrak{J} \bar{r}_{\beta'}) \cdot (\bar{r}_{\beta'} \mathfrak{J} 1)]\breve{q}\,\mathrm{Q}.$$

The second proposition of substantive logical possibility is that whatever is true of *some* of a class is true of the whole of *some* class. That is, if we accept a proposition of the form $\Sigma_i\, a_i \cdot b_i$, we can write

$$\Sigma_\gamma\, \Pi_i\, \bar{q}_{\gamma i} + \bar{a}_i + b_i,$$

though this will generally fail positively to assert, in itself, what is implied, that the collection j excludes whatever is a but not b, and includes something in common with a. There are, however, cases in which this implication is easily made plain.

Applying these two principles to the relation of correspondence, we get a new statement of the assertion that for every P there is a Q. Namely, if we write $a_{\alpha i}$ to signify that i is a relate of the relative r_α to some correlate, that is if $a_{\alpha i} = (i \prec r_\alpha \infty)$, if we write $b_{\alpha j}$ to signify that j is a correlate of the relative r_α to some relate, that is if $b_{\alpha j} = (j \prec \breve{r}_\alpha \infty)$, and if we write $p_{c\alpha}$ to signify that r_α is an aggregate of the relative r_c, that is, if $p_{c\alpha} = (r_\alpha \prec r_c)$, then the proposition that for every subject of P there is a subject of Q may be put in the form,

$$\Sigma_c \Sigma_\gamma \Pi_x \Pi_y \Sigma_\delta \Sigma_\epsilon \Pi_\alpha \Sigma_i \Sigma_j \Pi_\beta \Pi_u \Pi_v$$
$$[\bar{p}_{c\alpha} + a_{\alpha i} \cdot q_{\mathrm{P}i} \cdot b_{\alpha j} \cdot q_{\mathrm{Q}j} \cdot q_{\gamma j} \cdot (\bar{a}_{\alpha u} + 1_{iu}) \cdot (\bar{b}_{\alpha v} + 1_{jv}) \cdot (\bar{p}_{c\beta} + 1_{\alpha\beta} + \bar{a}_{\beta i} \cdot$$
$$b_{\beta j})] \cdot (\bar{q}_{\mathrm{P}x} + a_{\delta x} \cdot p_{c\delta}) \cdot (\bar{q}_{\mathrm{Q}y} + \bar{q}_{\gamma y} + b_{\epsilon y} \cdot p_{c\epsilon}).$$

This states that there is a collection of pairs, c, any single pair of which, α, has for its sole first subject a subject of P, and for its sole second subject a subject of Q which is at the same time a subject of a collection, j, and that no two pairs of the collection, c, have the same first subject or the same second subject, and that every subject of P is a first subject of some pair of this collection, c, and every subject of Q which is at the same time a subject of γ is a second subject of some pair of the same collection, c.

The third proposition of the doctrine of substantive logical possibility of which we have need is that all hecceities are alike in respect to their capacity for entering into possible pairs. Consequently, all the objects of any collection whatever may be severally and distinctly paired with all the objects of a collection which shall either be wholly contained in, or else shall entirely contain, any other collection whatever. Consequently,

$$\Pi_{\mathrm{P}} \Pi_{\mathrm{Q}} \Sigma_c \Sigma_\delta \Pi_x \Sigma_\delta \Pi_y \Sigma_\delta \Pi_\alpha \Sigma_i \Sigma_j \Pi_n \Pi_v \Pi_\beta \Pi_m \Pi_n$$
$$[\bar{p}_{c\alpha} + a_{\alpha i} \cdot q_{\mathrm{P}i} \cdot b_{\alpha j} \cdot q_{\delta j} \cdot (\bar{a}_{\alpha u} + 1_{uu}) \cdot (b_{\alpha v} + 1_{vj}) \cdot (\bar{p}_{c\beta} + 1_{\alpha\beta} + \bar{a}_{\beta i} \cdot$$
$$b_{\beta j})] \cdot (\bar{q}_{px} + a_{\delta x} \cdot p_{c\delta}) \cdot (\bar{q}_{\delta y} + b_{\epsilon y} \cdot p_{c\epsilon}) \cdot (\bar{q}_{\delta m} + q_{\mathrm{Q}m} + \bar{q}_{\mathrm{Q}n} + q_{\delta n}).$$

Although the above three propositions belong to a system of doctrine not universally recognised, yet I believe their truth is unquestionable. Suppose, now, that it is not true that for every subject of P there is a subject of Q. Then, in the last formula, $\Pi_m \bar{q}_{\delta m} \dotplus q_{Qm} \prec 0$. This leaves for the last factor $\Pi_n \bar{q}_{Qn} \dotplus q_{\delta n}$, and then the formula expresses that for every subject of Q there is a subject of P. In other words, we have demonstrated the important proposition that *two collections cannot be disparate in respect to correspondence*, but that for every subject of the one there must be a subject of the other.

The theorem $c \prec c \mathbin{\jmath} c$ is now established; for since of any two collections one corresponds to the other, we have $\infty \prec c \dotplus \check{c}$ or (non-relatively multiplying by $\check{c}$) $\check{c} \prec c$. Hence, $c \prec 1 c \prec (\check{c} \mathbin{\jmath} c)$ $c \prec \check{c} \mathbin{\jmath} c c \prec c \mathbin{\jmath} c c$; and, by the transitive principle $c c \prec c$, we finally obtain $c \prec c \mathbin{\jmath} c$.

Thus is established the conception of *multitude*. Namely, if for every subject of P there is a subject of Q, while there is not for every subject of Q a subject of P, the *multitude* of Q is said to be *greater* than that of P. But if for every subject of each collection there is a subject of the other, the *multitudes* of the two collections are said to be *equal* the one to the other. We may create a scale of objects, one for every group of equal collections. Calling these objects *arithms*, the first arithm will belong to 0 considered as a collection, the second to individuals, etc. Calling a collection the counting of which can be completed an *enumerable* collection, the multitude of any enumerable collection equals that of the arithms that precede its arithm. Calling a collection whose multitude equals that of all the arithms of enumerable collection a *denumerable* collection (because its subjects can all be distinguished by ordinal numbers, though the counting of it cannot be completed), the arithms preceding the arithm of denumerable collections form a denumerable collection. More multitudinous collections are greater than the collections of arithms which precede their arithm.

Let there be a denumerable collection, say the cardinal numbers; and let there be two houses. Let there be a collection of children, each of whom wishes to have those numbers placed in some way into those houses, no two children wishing for the same distribution, but every distribution being wished for by some child. Then, as Dr. George Cantor has proved, the collection of children is greater in multitude than the collection of numbers. Let a collection equal in multitude to that collection of children be called an *abnumeral* collection *of the first dignity*. The real numbers (surd and rational) constitute such a collection.

I now ask, suppose that for every way of placing the subjects of one collection in two houses, there is a way of placing the subjects of another collection in two houses, does it follow that for every subject of the former collection there is a subject of the latter? In order to answer this, I first ask whether the multitude of possible ways of placing the subjects of a collection in two houses can equal the multitude of those subjects. If so, let there be such a multitude of children. Then, each having but one wish, they can among them wish for every possible distribution of themselves among two houses. Then, however they may actually be distributed, some child will be perfectly contented. But ask each child which house he wishes himself to be in, and put every child in the house where he does not want to be. Then, no child would be content. Consequently, it is absurd to suppose that any collection can equal in multitude the possible ways of distributing its subjects in two houses.

Accordingly, the multitude of ways of placing a collection of objects abnumeral of the first dignity into two houses is still greater in multitude than that multitude, and may be called abnumeral of the second dignity. There will be a denumerable succession of such dignities. But there cannot be any multitude of an infinite dignity; for if there were, the multitude of ways of distributing it into two houses would be no greater than itself.[1]

[1] Inasmuch as the above theorem is, as I believe, quite opposed to the opinion prevalent among students of Cantor, and they may suspect that some fallacy lurks in the reasoning about wishes, I shall here give a second proof of a part of the theorem, namely that there is an endless succession of infinite multitudes related to one another as above stated, a relation entirely different, by the way, from those of the orders of infinity used in the calculus. I shall not be able to prove by this

We thus not only answer the question proposed, and show that of two unequal multitudes the multitude of ways of distributing the greater is the greater; but we obtain the entire scale of collectional

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second method, as is proved in the text, that there are no higher multitudes, and in particular no maximum multitude.

The ways of distributing a collection into two houses are equal to the possible combinations of members of that collection (including zero); for these combinations are simply the aggregates of individuals put into either one of the houses in the different modes of distribution. Hence, the proposition is that the combinations of whole numbers are more multitudinous than the whole numbers, that the combinations of combinations of whole numbers are still more multitudinous, the combinations of combinations of combinations again more multitudinous, and so on without end.

I assume the previously proved proposition that of any two collections there is one which can be placed in one-to-one correspondence with a part or the whole of the other. This obviously amounts to saying that the members of any collection can be arranged in a linear series such that of any two different members one comes later in the series than the other.

A part may be equal to the whole; as the even numbers are equal in multitude to all the numbers (since every number has a double distinct from the doubles of all other numbers, and that double is an even number). Hence, it does not follow that because one collection can be placed in one-to-one correspondence to a part of another, it is less than that other, that is, that it cannot also, by a rearrangement, be placed in one-to-one correspondence with the whole. This makes an inconvenience in reasoning which can be overcome in a manner I proceed to describe.

Let a collection be arranged in a linear series. Then, let us speak of a *section* of that series, meaning the aggregate of all the members which are later than (or as late as) one *assignable* member and at the same time earlier than (or as early as) a second *assignable* member. Let us call a series *simple* if it cannot be severed into sections each equal in multitude to the whole. A series not simple itself may be conceivably severed into *simple sections*, or it may be so arranged that it cannot be so severed (for example the series of rational fractions arranged in the order of their magnitudes). But suppose two collections to be each ranged in a linear series, and suppose one of them, A, is in one-to-one correspondence with a part of the other B. If now the latter series, B, can be severed into simple sections, in each of which it is possible to find a member at least as early in the series as any member of that section that is in correspondence with a member of the other collection A, and also a member at least as late in the series as any member of that section that is in correspondence with any member of the other collection, and if it is also possible to find a section of the series, B, equal to the whole series, B, in which it is possible to find a member *later* than any member that is in correspondence with any member of the collection, A, then I say that the collection, B, is greater than the collection, A. This is so obvious that I think the demonstration may be omitted.

Now, imagine two infinite collections, the α's and the β's, of which the β's are the more multitudinous. I propose to prove that the possible combinations of β's are more multitudinous than the possible combinations of α's. For let the pairs of conjugate combinations (meaning by conjugate combinations a pair each of which includes every member of the whole collection which the other excludes) of the β's be arranged in a linear series; and those of the α's in another linear series. Let the order of the pairs in each of the two series be subject to the rule that if of two pairs one contains a combination composed of fewer members than either combination of the other pair, it shall precede the latter in the series. Let the order of the pairs in the series of pairs of combinations of β's be further determined by the rule that where the first rule does not decide, one of two pairs shall precede the other whose smaller combination (this rule not applying where one combinations are equal) contains fewer β's which are in correspondence with α's in one fixed correspondence of all the α's with a part of the β's.

In this fixed correspondence each α has its β, while there is an infinitely greater multitude of β's without α's than with. Let the two series of pairs of combinations quantity, which we find to consist of two equal parts (that is two parts whose multitudes of grades are equal), the one finite, the other infinite. Corresponding to the multitude of 0 on the finite scale is the abnumeral of 0 dignity, which is the denumerable, on the infinite scale, etc.

So much of the general logical doctrine of quantity has been here given, in order to illustrate the power of the logic of relatives in enabling us to treat with unerring confidence the most difficult conceptions, before which mathematicians have heretofore shrunk appalled.

I had been desirous of examining Professor Schröder's developments concerning individuals and individual pairs; but owing to the length this paper has already reached, I must remit that to some future occasion.

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be so placed in correspondence that every pair of unequal combinations of α's is placed in correspondence with that pair of combinations of β's of which the smaller contains only the β's corresponding in the fixed correspondence to the smaller combination of α's; and let every pair of equal combinations of α's be put into correspondence with a pair of β's of which the smaller contains only the β's belonging in the fixed correspondence to one of the combinations of α's.

Then it is evident that each series will generally consist of an infinite multitude of simple sections. In none of these will the combinations be more multitudinous than those of the β's. In some, the combinations of α's will be equal to those of the β's; but in an infinitely greater multitude of such simple sections and each of these infinitely more multitudinous, the combinations of β's will be infinitely more multitudinous than those of the α's. Hence it is evident that the combinations of the β's will on the whole be infinitely more multitudinous than those of the α's.

That is if the multitude of finite numbers be a, and $2^a = b$, $2^b = c$, $2^c = d$, etc $a < b < c < d <$ etc. *ad infinitum.*

It may be remarked that the *finite* combinations of finite whole numbers form no larger a multitude than the finite whole numbers themselves. But there are nfinite collections of finite whole numbers; and it is these which are infinitely more numerous than those numbers themselves.